Queuing Formulas

Suppose that customers enter a bank at random times and must be served by a single teller. We will model the customers' entrance times by a Poisson process with parameter λ and the teller's time serving a customer by a Poisson process with parameter μ, where $\lambda < \mu$.

Let p_n be the long term probability that there are n customers in the bank, either with the teller or in line. Then let

$$\begin{aligned} I_0 &= [t, t + \mathrm{d}t) \\ I_1 &= [t + \mathrm{d}t, t + 2\,\mathrm{d}t). \end{aligned}$$

where t is a randomly chosen time. Then we have

$$\begin{array}{rclclcl} \mathrm{P}(n = 0 \text{ in } I_1) &=& \mathrm{P}(n = 0 \text{ in } I_0) & \mathrm{P}(0 \text{ customers enter}) &+& \mathrm{P}(n = 1 \text{ in } I_0) & \mathrm{P}(1 \text{ customer served}) \\ p_0 &=& p_0 \cdot & (1 - \lambda\,\mathrm{d}t) &+& p_1 \cdot & \mu\,\mathrm{d}t \end{array}$$

from which we get

$$p_1 = \tfrac{\lambda}{\mu} p_0.$$

Similarly we have

$$p_1 = p_2 \cdot \mu\,\mathrm{d}t + p_1 \cdot (1 - (\lambda + \mu)\,\mathrm{d}t) + p_0 \cdot \lambda\,\mathrm{d}t$$

because $(\lambda + \mu)\,\mathrm{d}t$ is the probability that either the teller finishes with a customer or a new customer enters the bank during I_0. This yields a similar result, namely

$$p_2 = \tfrac{\lambda}{\mu} p_1.$$

In fact the algebra is the same for any n, so we get

$$\begin{aligned} \text{for all n,} \quad p_{n+1} &= \tfrac{\lambda}{\mu} p_n \\ p_n &= \left(\tfrac{\lambda}{\mu}\right)^n p_0. \end{aligned}$$

Since the p_n are probabilities their sum must be 1.

$$1 = \sum_{n=0}^{\infty} p_n = p_0 \frac{1}{1 - \frac{\lambda}{\mu}}$$

so that

$$p_0 = \tfrac{\mu - \lambda}{\mu} = 1 - \tfrac{\lambda}{\mu}.$$

So the general formula for p_n is

$$p_n = (\tfrac{\lambda}{\mu})^n (1 - \tfrac{\lambda}{\mu}).$$

Now let W be the total time a customer spends in the waiting line before seeing the teller. Since the teller takes an average of $1/\mu$ minutes to serve each customer, we get

$$\begin{aligned} \mathrm{E}\,(W \mid n = 0) &= 0 \\ \mathrm{E}\,(W \mid n = 1) &= \tfrac{1}{\mu} \\ \mathrm{E}\,(W \mid n = 2) &= \tfrac{2}{\mu} \\ &\vdots \\ \mathrm{E}\,(W \mid n) &= \tfrac{n}{\mu}. \end{aligned}$$

Therefore the overall average time spent by a customer in the waiting line is

$$
\begin{aligned}
\mathrm{E}(W) &= \sum_{n=0}^{\infty} p_n \frac{n}{\mu} \\
&= \frac{1}{\mu} \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) \\
&= \frac{1}{\mu} \frac{\lambda}{\mu} \left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^{n-1} \\
&= \frac{\lambda}{\mu^2} \left(1 - \frac{\lambda}{\mu}\right) \frac{1}{\left(1 - \frac{\lambda}{\mu}\right)^2} \\
&= \frac{\lambda}{\mu} \frac{1}{\mu - \lambda}.
\end{aligned}
$$

This means that the average total time spent in the system (line and teller) is

$$
\begin{aligned}
& \frac{\lambda}{\mu} \frac{1}{\mu - \lambda} + \frac{1}{\mu} \\
= & \frac{\lambda + \mu - \lambda}{\mu(\mu - \lambda)} \\
= & \frac{1}{\mu - \lambda}.
\end{aligned}
$$

So we have shown the following.

$$
p_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) \tag{1}
$$

$$
\text{average time in line} = \frac{\lambda}{\mu} \cdot \frac{1}{\mu - \lambda} \tag{2}
$$

$$
\text{average time in bank} = \frac{1}{\mu - \lambda} \tag{3}
$$