Chapter 3:

EX: (3.1) Suppose $X(t), t \geq 0$ is a B.M. with drift μ and variance σ^2 and that $X(0) = 0$. We want to show that $-X(t), t \geq 0$ is a B.M. with drift $-\mu$ and variance σ^2.

To show that a stochastic process is a B.M., we need to show two properties:

(a) $X(0)$ is a given constant

Since, by assumption, $X(0) = 0$, it follows:

$-X(0) = -0 = 0$, and hence, $-X(0)$ is a given costant (zero).

(b) $\forall s,t$: $X(s+t) - X(t) \sim \text{Normal}(\mu \cdot s, s \cdot \sigma^2)$.

Let $y, t \in \mathbb{R}$: Consider $-X(y+t) - [-X(t)] = X(t) - X(y+t)$

$= -[X(y+t) - X(t)]$

But, by assumption $X(y+t) - X(t) \sim \text{Normal}(\mu \cdot y, y \cdot \sigma^2)$.

Hence, setting $Y = -[X(y+t) - X(t)]$, by exercise 2.4., (setting $a=0$ & $b=-1$) we know that $Y \sim$ Normal with:

$E[Y] = E[-[X(y+t) - X(t)]] = -E[[X(y+t) - X(t)]] = -\mu$ and

$Var[Y] = Var[-[X(y+t) - X(t)]] = (-1)^2 Var[X(y+t) - X(t)] = \sigma^2$.

Since (a) & (b) hold, it follows that $-X(t), t \geq 0$ is a B.M. with drift $-\mu$ and variance σ^2.

Ex: (3.2) Let $X(t), t \geq 0$ be a B.M. with drift $\mu = 3$ and $\sigma^2 = 9$.

If $X(0) = 10$, then:

(a) $E[X(2)] = E[X(2) - X(0) + X(0)]$

$= E[X(2) - X(0)] + E[X(0)]$

But $X(2) - X(0) \sim \text{Normal}((2-0) \cdot 3 = 6, (2-0) \cdot 9 = 18)$, hence

$= 6 + 10$

$= \boxed{16}$

(b) $Var(X(2)) = Var(X(2)-X(0)+X(0))$

AND we know $X(2)-X(0) \sim Normal((2-0)\cdot 3 = 6,\ (2-0)9 = 18)$, Moreover, $X(2)-X(0)$ is independent of $X(0)=10$, hence

$$= Var(X(2)-X(0)) + Var(X(0))$$

$= 18 + 0$, since the variance of a constant is 0

$$= \boxed{18}$$

(c) $P(X(2)>20) = P(X(2)-X(0)+X(0)>20)$

$$= P(X(2)-X(0) > 20 - X(0))$$

$$= P(X(2)-X(0) > 20-10)$$

$$= P(X(2)-X(0) > 10),$$

where $Y = X(2)-X(0) \sim Normal(6, 18)$

$$= P\left(\frac{Y-6}{\sqrt{18}} > \frac{10-6}{\sqrt{18}}\right)$$

$$= P\left(X_{0,1} > \frac{4}{\sqrt{18}}\right) = 1 - P\left(X_{0,1} \leq \frac{4}{3\sqrt{2}}\right) = 1 - \phi\left(\frac{4}{3\sqrt{2}}\right) \simeq \boxed{0.173}$$

(d) $P(X(.5)>10) = P(X(.5)-X(0)+X(0)>10)$

$$= P(X(.5)-X(0) > 10 - X(0))$$

$$= P(X(.5)-X(0) > 10-10)$$

$$= P(X(.5)-X(0) > 0)$$

where $Y = X(.5)-X(0) \sim Normal((.5-0)\cdot 3, (.5-0)\cdot 9) = Normal(\underset{\mu}{\frac{3}{2}}, \frac{9}{2} = \sigma^2)$

$$= P\left(\frac{Y-\frac{3}{2}}{\sqrt{\frac{9}{2}}} > \frac{0-\frac{3}{2}}{\sqrt{\frac{9}{2}}}\right)$$

$$= P\left(X_{0,1} > \frac{-\frac{3}{2}}{\frac{3}{\sqrt{2}}}\right) = 1 - P\left(X_{0,1} \leq -\frac{\sqrt{2}}{2}\right)$$

$$= 1 - \phi\left(-\frac{\sqrt{2}}{2}\right)$$

$$\simeq \boxed{0.760}$$

Ex (3.3) Let $\Delta = 0.1$ in the approximation model to B.M in (3.2).

(a) $$E[X(1)] = E[X(1)-X(0)+X(0)]$$
$$= E[X(1)-X(0)] + E[X(0)]$$
$$= E\left[3\sqrt{0.1}\sum_{i=1}^{1/0.1} X_i\right] + 10$$
$$= 3\sqrt{0.1}\sum_{i=1}^{10} E[X_i] + 10$$
$$= 3\sqrt{0.1}\cdot 10 \cdot \frac{\cancel{3}}{\cancel{3}}\sqrt{0.1} + 10$$
$$= 3\cdot 10\cdot 0.1 + 10$$
$$= 10(3\times 0.1 + 1)$$
$$= 10(1.3)$$
$$= \boxed{13}$$

(b) $$Var(X(1)) = Var(X(1)-X(0)+X(0))$$
$$= Var(X(1)-X(0))$$
$$= 9\cdot 1[1-(2p-1)^2]$$, where $2p-1 = 2\cdot\left[\frac{1}{2}\left(1+\frac{\mu}{6}\sqrt{\Delta}\right)\right]-1 = \frac{\mu}{6}\sqrt{\Delta}$
$$= 9[1-(\sqrt{0.1})^2]$$
$$= 9[1-0.1]$$
$$= 9[.9]$$
$$= \boxed{8.1}$$

(c) $$P(X(.5) > 10) = P(X(.5)-X(0)+X(0) > 10)$$
$$= P(X(.5)-X(0) > 0)$$

Ex: (3.6) Let $S(t), t \geq 0$ be a geometric Brownian motion with drift μ and volatility parameter σ. Assuming that $S(0) = s$, find $Var(S(t))$.

We begin:
$$\begin{aligned}
Var(S(t)) &= Var(e^{X(t)}) \quad \cdots \text{ by definition of } S(t) \\
&= Var(e^{X(t)-X(0)+X(0)}) \quad \cdots \text{ arithmetic} \\
&= Var(e^{X(0)} e^{X(t)-X(0)}) \quad \cdots \text{ properties of exponential} \\
&= [e^{X(0)}]^2 \, Var(e^{X(t)-X(0)}) \quad \cdots \text{ properties of variance} \\
&= [S(0)]^2 \, Var(e^{X(t)-X(0)}) \quad \cdots \text{ by definition of } S(t) \\
&= s^2 \, Var(e^{Y(t)}) \quad \cdots \text{ letting } Y(t) = X(t) - X(0)
\end{aligned}$$

Now, we know that, since $X(t), t \geq 0$ is a B.M., with μ and σ^2, then
$$Y(t) = X(t) - X(0) \sim Normal(\mu t, \sigma^2 t).$$

We also know $Var(X) = E[X^2] - E[X]^2$. Therefore,
$$Var(S(t)) = s^2 \, Var(e^{Y(t)}) = s^2 \left\{ \underbrace{E[e^{2Y(t)}]}_{A} - \underbrace{E[e^{Y(t)}]^2}_{B} \right\}$$

Let us solve A and B separately:

A $= E[e^{2Y(t)}]$, for this we need that, if $Y \sim Normal(\mu t, \sigma^2 t)$, then $2Y \sim Normal(2\mu t, 4\sigma^2 t)$; AND we also need that: if $2Y \sim Normal(2\mu t, 4\sigma^2 t)$, then $E[e^{2Y(t)}] = \exp\{E[2Y] - Var(2Y)/2\}$

Hence,
$$\begin{aligned}
A = E[e^{2Y(t)}] &= \exp\{E[2Y] + Var(2Y)/2\} \\
&= e^{2\mu t + 4\sigma^2 t/2} \\
&= e^{2\mu t + 2\sigma^2 t} \\
&= e^{2t(\mu + \sigma^2)}
\end{aligned}$$

(B) $= E[e^{Y(t)}]^2 \Rightarrow$ using some facts as in (A)

$$= \exp\{E[Y(t)] + Var[Y(t)]/2\}^2$$

$$= e^{2(\mu t + \sigma^2 t/2)}$$

$$= e^{2t(\mu + \sigma^2/2)}$$

Finally, $Var(S(t)) = s^2\left\{e^{2t(\mu+\sigma^2)} - e^{2t(\mu+\sigma^2/2)}\right\}$

$$\boxed{Var(S(t)) = s^2 e^{2\mu t + t\sigma^2}\left(e^{t\sigma^2} - 1\right)}$$

Ex (3.8) Let $S(v), v \geq 0$ be a geometric Brownian motion process with drift μ and volatility parameter σ, having $S(0) = s$.
Find $P(\max_{0\leq v\leq t} S(v) \geq y)$.

The event that $\max_{0\leq v\leq t} S(v) \geq y$ is equivalent, by definition, to the event that $\max_{0\leq v\leq t} s e^{X(v)} \geq y$. Now, since the logarithm is an increasing function, this last event is equivalent to:
$\max_{0\leq v\leq t} X(v) \geq \log(y/s)$. Therefore; for $s > 0$:

$$P(\max_{0\leq v\leq t} S(v) \geq y) = P(\max_{0\leq v\leq t} X(v) \geq \log(y/s))$$

But we know the distribution of $\max_{0\leq v\leq t} X(v) = M(t)$, from Corollary 3.4.1

Hence, $\boxed{P(M(t) \geq \log(y/s)) = e^{2\log(y/s)\mu/\sigma^2}\left(1 - \phi\left\{\frac{\mu t + \log(y/s)}{\sigma\sqrt{t}}\right\}\right) + \left(1 - \phi\left\{\frac{\log(y/s) - \mu t}{\sigma\sqrt{t}}\right\}\right)}$

This is just Corollary 3.4.1, setting $y \mapsto \log(y/s)$

EX: (3.7). Let $\{X(t), t \geq 0\}$ be a B.M. with drift μ and variance σ^2. Assume that $X(0)=0$, and let T_y be the first time that the process is equal to y. Then, for $y>0$:

$$P(T_y < \infty) = \lim_{t\to\infty} P(T_y < t)$$

$$= \lim_{t\to\infty} e^{2y\mu/\sigma^2}\,\overline{\Phi}\left(\frac{y+\mu t}{\sigma\sqrt{t}}\right) + \overline{\Phi}\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) \quad \ldots \text{ By Corollary 3.4.1.}$$

$$= e^{2y\mu/\sigma^2} \lim_{t\to\infty} \overline{\Phi}\left(\frac{y+\mu t}{\sigma\sqrt{t}}\right) + \lim_{t\to\infty} \overline{\Phi}\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) \quad \ldots \text{ By Properties of limit.}$$

To solve this, note that:

$$\lim_{t\to\infty} \frac{y+\mu t}{\sigma\sqrt{t}} = \lim_{t\to\infty}\left(\frac{y}{\sigma\sqrt{t}} + \frac{\mu t}{\sigma\sqrt{t}}\right) \quad \ldots \text{ arithmetic}$$

$$= \lim_{t\to\infty} \frac{y}{\sigma\sqrt{t}} + \lim_{t\to\infty} \frac{\mu t}{\sigma\sqrt{t}} \quad \ldots \text{ By Properties of limit.}$$

$$= 0 + \frac{\mu}{\sigma} \lim_{t\to\infty} \sqrt{t}$$

$$= \frac{\mu}{\sigma}\infty$$

So whether we approach $+\infty$ or $-\infty$ depends on the sign of $\frac{\mu}{\sigma}$. assuming $\sigma>0$, we then have:

If $\mu>0$, then $\lim_{t\to\infty} \overline{\Phi}\left(\frac{y+\mu t}{\sigma\sqrt{t}}\right) = \overline{\Phi}(+\infty) = 0$ AND

(what if $\mu=0$) $\lim_{t\to\infty} \overline{\Phi}\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) = \overline{\Phi}(-\infty) = 1$

If $\mu<0$, then $\lim_{t\to\infty} \overline{\Phi}\left(\frac{y+\mu t}{\sigma\sqrt{t}}\right) = \overline{\Phi}(-\infty) = 1$ AND

$\lim_{t\to\infty} \overline{\Phi}\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) = \overline{\Phi}(+\infty) = 0$.

Therefore

$$P(T_y<\infty) = \begin{cases} 1 & \text{if } \mu>0 \\ e^{2y\mu/\sigma^2} & \text{if } \mu<0 \end{cases}$$

Let $M = \max_{0<t<\infty} X(t)$ be the maximal value ever attained by the process. then, since the events

$$M(t) \geq y \iff T_y \leq t \quad \text{are equivalent,}$$

we must have: (If $\mu < 0$)

$$P(M > y) = P(T_y < \infty)$$
$$= e^{-2y\mu/\sigma^2} \quad \ldots \text{ by previous part.}$$

AND so

$$P(M \leq y) = 1 - P(M > y) = 1 - e^{-2y\mu/\sigma^2},$$

which shows that $M \sim Exp(\lambda = -2\mu/\sigma^2)$.

(#7) Assuming that $\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}x^2}\,dx = 1$, SHOW that

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}x^2+ax+b}\,dx = e^{\frac{1}{2}a^2+b}.$$

USE THIS to derive the formula in the third remark of Section 3.3, i.e., $E[S(t)] = s\,e^{\mu t + t\sigma^2/2}$

Sol: the key point here is to perform the following factorization:

$$-\frac{1}{2}x^2 + ax + b = -\frac{1}{2}(x-a)^2 + \left(\frac{1}{2}a^2 + b\right). \quad \text{Then,}$$

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}x^2+ax+b}\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}(x-a)^2+(\frac{1}{2}a^2+b)}\,dx \quad \ldots \text{ by previous factorization}$$

$$= \frac{e^{\frac{1}{2}a^2+b}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}(x-a)^2}\,dx \quad \ldots \text{ by properties of exponential function}$$

Now, make the change of variables: $x - a = y \Rightarrow dx = dy$, to obtain,

$$= \frac{e^{\frac{1}{2}a^2+b}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}y^2}\,dy \overset{\text{by hypothesis}}{=} \frac{e^{\frac{1}{2}a^2+b}}{\sqrt{2\pi}}\cdot\sqrt{2\pi} = e^{\frac{1}{2}a^2+b}$$

Showing that: $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2+ax+b} dx = e^{-\frac{1}{2}a^2+b}$.

Now, let us compute: $E[S(t)]$, assuming $S(0)=S$.

By definition: $E[S(t)] = E[s e^{X(t)}]$, where we know $X(t)$ is a B.M. where $X(0)=0$.

$X(t) \sim \text{Normal}(\mu t, \sigma^2 t)$, i.e.,

then,

$E[S(t)] = E[s e^{X(t)}]$ .... By definition

$= s\, E[e^{X(t)}]$ .... linearity of expectation

$= s \int_{\mathbb{R}} e^{x} \cdot \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}} dx$ .... pdf of a normal$(\mu t, \sigma^2 t)$

$= \frac{s}{\sqrt{2\pi t}\,\sigma} \cdot \int_{\mathbb{R}} e^{\left\{-\frac{(x-\mu t)^2}{2\sigma^2 t} + x\right\}} dx$ ... rearranging terms

Now, make the change: $x - \mu t = \sigma\sqrt{t}\, y \Rightarrow \begin{cases} dx = \sigma\sqrt{t}\, dy \\ x = \sigma\sqrt{t}\, y + \mu t \end{cases}$

$= \frac{s}{\sqrt{2\pi t}\,\sigma} \int_{\mathbb{R}} e^{-\frac{(\sigma\sqrt{t}\, y)^2}{2\sigma^2 t} + \sigma\sqrt{t}\, y + \mu t} (\sigma\sqrt{t})\, dy$ .... making the change

$= \frac{s}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}y^2 + \sigma\sqrt{t}\, y + \mu t} dy$ .... simplifying

$= \frac{s}{\sqrt{2\pi}} \left(e^{\frac{1}{2}(\sigma\sqrt{t})^2 + \mu t}\right)\sqrt{2\pi}$ .... By previous result

$= \boxed{s\, e^{\frac{1}{2}\sigma^2 t + \mu t}}$