Chapter 4:

Ex: (4.1). What is the effective interest rate when the nominal interest rate of 10 % is

(a) compounded semiannually:

$$r_{eff} = \left(1 + \frac{0.1}{2}\right)^2 - 1 = \cancel{1} + 0.1 + 0.05^2 - \cancel{1} = 0.1 + 0.0025 = 0.1025,$$

So, the effective interest rate is $\boxed{10.25\%}$

(b) compounded quartely:

$$r_{eff} = \left(1 + \frac{0.1}{4}\right)^4 - 1 = 1.025^4 - 1 \approx 0.1038,$$

So, the effective interest rate is $\boxed{10.38\%}$

(c) compounded continuously:

$$r_{eff} = \lim_{n \to \infty} \left(1 + \frac{0.1}{n}\right)^n - 1 = e^{0.1} - 1 \approx 0.1052,$$

So, The effective interest rate is $\boxed{10.52\%}$

Ex: (4.4) Let P be my current funds. Suppose I receive interest rate r compounded yearly. Then, after T years I would have $P(1+r)^T$ funds. We want to find T such that:

$$\cancel{P}(1+r)^T = 3\cancel{P} \iff (1+r)^T = 3$$

We can find the exact solution: $\iff T\ln(1+r) = \ln(3)$

$$\iff \boxed{T = \frac{\ln(3)}{\ln(1+r)}}$$

OR, we could use the approximation $e^x \approx 1 + x$, for small x, (say $|x| < 1$). Then,

$$e^{rt} \approx (1+r)^T = 3 \implies e^{rT} \approx 3 \implies rT \approx \ln(3) \implies T \approx \frac{\ln(3)}{r}$$

Further using $\ln(3) \approx 1.1$, we have an approximate formula:

$\boxed{T \approx \frac{1.1}{r}}$, which would make sense since we usually think of r as being small $r < 1$.

Ex 4.9: the purchase is for \$4,200. there is a down payment of \$1,000, which means that the amount borrowed B is:

$$B = \$4{,}200 - \$1{,}000 = \$3{,}200.$$

There will be 24 payments of \$160, beginning one month from the time of the purchase. the effective interest rate r, is the value for which:

$$PV(B) = \$3{,}200 = \overbrace{\$160\,\alpha + \$160\,\alpha^2 + \cdots + \$160\,\alpha^{24}}^{\text{today's dollars}}$$
$$= \$160 \sum_{i=1}^{24} \alpha^i, \quad \text{where } \alpha = \frac{1}{1+r}$$

Solving this equation:

$$\$3{,}200 = \$160 \sum_{i=1}^{24} \alpha^i \iff \$20 = \sum_{i=1}^{24} \alpha^i = \alpha \sum_{i=0}^{23} \alpha^i = \frac{\alpha(1-\alpha^{24})}{1-\alpha}$$

$$\Rightarrow 20 = \frac{\alpha(1-\alpha^{24})}{1-\alpha} = \frac{\left[\frac{1}{1+r}\right]\left[1-\left[\frac{1}{1+r}\right]^{24}\right]}{1-\frac{1}{1+r}} = \frac{\left[\frac{1}{\cancel{1+r}}\right]\left[1-\frac{1}{(1+r)^{24}}\right]}{\frac{r}{\cancel{1+r}}}$$

$$\Rightarrow 20 = \frac{1-\frac{1}{(1+r)^{24}}}{r} \Rightarrow 20r = 1-\frac{1}{(1+r)^{24}} \Rightarrow 1-20r = \frac{1}{(1+r)^{24}}$$

$$\Rightarrow \frac{1}{1-20r} = (1+r)^{24} \Rightarrow (1+r)^{24}(1-20r) - 1 = 0,$$

Solving this equation with an algebra system $\Rightarrow r \approx 0.015$

Hence, the effective interest rate is $\boxed{r_{eff} = 1.5\% \text{ per month}}$

Ex 4.14: STREAM:

years		1/2	1	1 1/2	2	2 1/2	3	3 1/2	4	4 1/2	5
payments	0	1	2	3	4	5	6	7	8	9	10
	−1,000	30	30	30	30	30	30	30	30	30	1030

5% interest rate compounded continuously.

Compounded continuously means rate $= \lim_{n\to\infty}\left\{1+\frac{\left[\frac{.05}{2}\right]}{n}\right\}^n = e^{0.025}$, since payments are semiannually.

then, $PV(\text{STREAM}) = -1{,}000 + 30\alpha + 30\alpha^2 + \cdots + 30\alpha^9 + 1030\alpha^{10}$

$$= -1{,}000 + \left[30\alpha \sum_{i=0}^{8} \alpha^i\right] + 1030\alpha^{10}, \quad \text{where } \alpha = \frac{1}{e^{0.025}}$$

$$= -1{,}000 + 30\alpha\left[\frac{1-\alpha^9}{1-\alpha}\right] + 1030\alpha^{10} \qquad \iff \alpha = e^{-0.025}$$

$$= -1{,}000 + 30e^{-0.025}\left[\frac{1-e^{-9*0.025}}{1-e^{-0.025}}\right] + 1030\, e^{-0.25}$$

$$= -1{,}000 + 30\frac{\left[e^{-0.025} - e^{-0.25}\right]}{1-e^{-0.025}} + 1030\, e^{-0.25}$$

$$= -1{,}000 + 1040.936 + 1004.57 = \boxed{\$\,40.936}$$

Ex (4.31) Borrow money @ 8% per year. (interest rat)
Save money @ 5% per year.

Consider the following yearly cash flows of an investment:

-1000, 900, 800, -1200, 700.

Should you invest?

I should invest if and only if I make money at the end.
Since to borrow money and save money there are different rates,
we should analyze this stream year by year:

YEAR 0: Borrow \$1000. BALANCE -\$1000

YEAR 1: Amount owed: -\$1,000 @ 8% = -\$1,080
Amount received: \$900
Balance: -\$1,080 + \$900 = -\$180

YEAR 2: Amount owed: -\$180 @ 8% = \$194.4
Amount received: \$800
Balance: -194.4 + 800 = \$605.6

YEAR 3: Amount saved: \$605.6 @ 5% = \$635.88
Amount received: -1200\$
BALANCE = 635.88 - 1200 = -\$564.12

YEAR 4: Amount owed: -\$564.12 @ 8% = -\$609.2496
Amount received: \$700
BALANCE = 700 - 609.2496 = $\boxed{\$\,90.7504}$ > 0 means the investment has a positive value

$\boxed{\text{YES, invest}}$

Ex: (4.33) Let $\bar{r}(t) = \frac{1}{t}\int_0^t r(s)\,ds$ be the yield curve, for $t>0$.

($\Rightarrow$) Suppose $\bar{r}(t)$ is a nondecreasing function of t, i.e.,

$\forall\, t_1, t_2 \geq 0$: If $t_1 \geq t_2$ then $\bar{r}(t_1) \geq \bar{r}(t_2)$.

Now, by definition, $P(\alpha t) = e^{-\int_0^{\alpha t} r(s)ds} = e^{-\alpha t\left[\frac{1}{\alpha t}\int_0^{\alpha t} r(s)\,ds\right]} = e^{-\alpha t \bar{r}(\alpha t)}$

AND, $P(t)^{\alpha} = \left[e^{-\int_0^t r(s)ds}\right]^{\alpha} = e^{-\alpha\int_0^t r(s)ds} = e^{-\alpha t\left[\frac{1}{t}\int_0^t r(s)ds\right]} = e^{-\alpha t \bar{r}(t)}$

Note now that $\alpha \in [0,1]$ and $t>0$, therefore $-\alpha t \leq 0$

Also, since $\alpha \in [0,1]$, $t \geq \alpha t$, but then, from hypothesis:

$\Rightarrow \bar{r}(t) \geq \bar{r}(\alpha t)$. Since e^{-x} is a (for $x>0$) decreasing function, we have

$$P(t)^{\alpha} = e^{-\alpha t \bar{r}(t)} \leq e^{-\alpha t \bar{r}(\alpha t)} = P(\alpha t)$$

Showing the result: $P(\alpha t) \geq P(t)^{\alpha}$

($\Leftarrow$) this direction follows inmediately from ($\Rightarrow$), reading it from bottom to top.

Ex: (4.8) STREAM:

YEARS		1/2	1	1 1/2	2	2 1/2	3	3 1/2	4	4 1/2	5
PAYOUT	-\$10,000	\$1500	\$1500	\$1500	\$1500	\$1500	\$1500	\$1500	\$1500	\$1500	\$1500 + \$10,000

$$PV(\text{STREAM}) = -\$10{,}000 + \left[\sum_{i=1}^{10} \$1500\,\alpha^{6i}\right] + \$10{,}000\,\alpha^{60},$$

where $\alpha = \frac{1}{1+r/12}$, monthly compound $\left[\text{so that } \alpha^{6i} = \left(\frac{1}{1+r/12}\right)^{6i} = \frac{1}{(1+\frac{r}{12})^{6i}}\right.$

So, for different values of r:

If $r = 0.06$ then $PV(\text{STREAM}) = \$1706.04057$

If $r = 0.10$ then $PV(\text{STREAM}) = \$0.$

If $r = 0.12$ then $PV(\text{STREAM}) = -\$736.00871$

These calculated on a computer.

Note that $r < 0.10$ in order for the investment to make sense.

#8. Consider two yearly income streams in dollars where the first payment is made immediately:

A: 100, 80, 211

B: 90, 100, 200

(a). For what interest rates $r \in \mathbb{R}$ is $PV(A) > PV(B)$?

By definition:

$PV(A): 100 + 80\alpha + 211\alpha^2$, where $\alpha = \frac{1}{1+r}$.

$PV(B): 90 + 100\alpha + 200\alpha^2$

If we try to solve $PV(A) = PV(B)$, we get:

$$PV(A) = PV(B) \iff 100 + 80\alpha + 211\alpha^2 = 90 + 100\alpha + 200\alpha^2$$
$$\iff 11\alpha^2 + (-20)\alpha + 10 = 0.$$

the discriminant of this equation is given b: $(-20)^2 - 4 \cdot 11 \cdot 10 = 400 - 440 = -40 < 0$, this means that there is no α (and hence, no $r \in \mathbb{R}$) s.t. $PV(A) = PV(B)$.

Since both $PV(A)$ and $PV(B)$ are quadratic polynomials in α, we conclude that either $PV(A) > PV(B)$ for all $r \in \mathbb{R}$ or that $PV(A) < PV(B)$ for all $r \in \mathbb{R}$.

Hence, it sufficies to check one value, say $\alpha = 0$, in which case $PV(A|\alpha=0) = 100 > 90 = PV(B|\alpha=0)$, to conclude:

$$PV(A) > PV(B) \text{ for any choice of } r \in \mathbb{R}.$$

(b) Repeat part (a), but with the last value of stream A changed to 209.

By definition:

$PV(A) = 100 + 80\alpha + 209\alpha^2$, where $\alpha = \frac{1}{1+r}$

$PV(B) = 90 + 100\alpha + 200\alpha^2$

In this case, there will be real solutions to:

$$PV(A) = PV(B) \iff 100 + 80\alpha + 209\alpha^2 = 90 + 100\alpha + 200\alpha^2, \text{ where } \alpha = \frac{1}{1+r}$$

$$\iff 9\alpha^2 - 20\alpha + 10 = 0$$

$$\iff \alpha = \frac{20 \pm \sqrt{400-360}}{2\cdot 9} = \frac{20 \pm \sqrt{40}}{2\cdot 9} = \frac{20 \pm 2\sqrt{10}}{2\cdot 9} = \frac{\cancel{2}(10 \pm \sqrt{10})}{\cancel{2}\cdot 9} = \frac{10 \pm \sqrt{10}}{9}$$

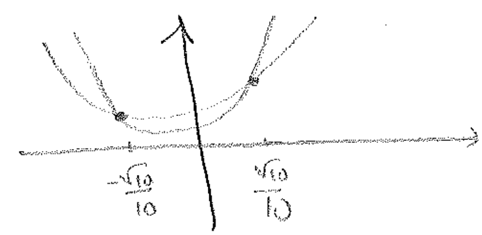
Hence, $\alpha_1 = \frac{10+\sqrt{10}}{9}$ and $\alpha_2 = \frac{10-\sqrt{10}}{9}$.

If $\alpha_1 = \frac{10+\sqrt{10}}{9} = \frac{1}{1+r_1}$, then, $r_1 = \frac{9}{10+\sqrt{10}} - 1 = \boxed{\frac{-\sqrt{10}}{10} = r_1}$

If $\alpha_2 = \frac{10-\sqrt{10}}{9} = \frac{1}{1+r_2}$, then, $r_2 = \frac{9}{10-\sqrt{10}} - 1 = \boxed{\frac{\sqrt{10}}{10} = r_2}$

So, the only values of r for which $PV(A) = PV(B)$ are r_1, r_2.

In pictures we have:



since both are quadratic polynomials

So, we need only to determine what happens in one of the 3 partitions: $(-\infty, -\frac{\sqrt{10}}{10})$, $[-\frac{\sqrt{10}}{10}, \frac{\sqrt{10}}{10}]$, $(\frac{\sqrt{10}}{10}, \infty)$, to know what happens in all of $\mathbb{R}$.

Take $r = 0 \in [-\frac{\sqrt{10}}{10}, \frac{\sqrt{10}}{10}]$. If $r=0$ then $\alpha = \frac{1}{1+0} = 1$. In this case,

$$PV(A) = 100 + 80 + 209 = 389 < 390 = 90 + 100 + 200 = PV(B)$$

Therefore, for $r \in [-\frac{\sqrt{10}}{10}, \frac{\sqrt{10}}{10}]$, we have $PV(A) < PV(B)$.

In all other cases $r \in \mathbb{R} \setminus [-\frac{\sqrt{10}}{10}, \frac{\sqrt{10}}{10}]$: $PV(A) > PV(B)$

#9. Suppose (continuously compounded) interest rates will remain at 3% from now (time 0) until time 3 years, and then jump instantly to 5% and stay there forever.

(a). What would you pay at time 0 for the right to receive \$1 in 4 years? (That is, what is $P(4)$?).

By definition:

$$P(4) = \frac{1}{D(4)} = \exp\left\{-\int_0^4 r(s)ds\right\},$$

but our function $r(s)$ is defined as:

$$r(s) = \begin{cases} 0.03 & \text{if } 0 \leq s \leq 3 \\ 0.05 & \text{if } s > 3 \end{cases}$$

So, we need to split the integral above as follows:

$$P(4) = \frac{1}{D(4)} = \exp\left\{-\int_0^4 r(s)\,ds\right\} = \exp\left\{-\left[\int_0^3 0.03\,ds + \int_3^4 0.05\,ds\right]\right\}$$

$$= \exp\{-[0.03(3-0) + 0.05(4-3)]\}$$

$$= \exp\{-[0.09 + 0.05]\}$$

$$= \exp\{-0.14\} \approx \boxed{0.8693582}\ \$$$

(b) Write a formula for $P(t)$ valid for every $t > 0$.

$$P(t) = \frac{1}{D(t)} = \exp\left\{-\int_0^t r(s)\,ds\right\} = \begin{cases} \exp\left\{-\int_0^t 0.03ds\right\} = e^{-0.03t} & \text{if } 0 \leq t \leq 3 \\ e^{-0.09}\, e^{\left\{-\int_3^t 0.05 ds\right\}} = e^{-0.09-0.05(t-3)} & \text{if } t > 3 \end{cases}$$

#10. Suppose that $P(t) = \frac{1}{1+e^t}$ for all $t >= 0$

(a) What is $D(t)$?

By definition: $P(t) = \frac{1}{D(t)} \Rightarrow \frac{1}{P(t)} = \boxed{D(t) = 1 + e^t}$ $\forall t >= 0$

(b) What is $r(s)$?

$$\frac{D'(s)}{D(s)} = r(s) \Rightarrow \frac{(1+e^s)'}{1+e^s} = r(s) \Rightarrow \boxed{r(s) = \frac{e^s}{1+e^s}} \quad \forall s \geq 0$$

(c) What is the yield curve?

By definition:

$$\bar{r}(t) = \frac{1}{t}\int_0^t r(s)\,ds = \frac{1}{t}\int_0^t \frac{e^s}{1+e^s}\,ds$$

$$= \frac{1}{t}\Big[\ln(1+e^s)\Big]_0^t$$

$$= \frac{1}{t}\Big[\ln(1+e^t) - \ln(1+e^0)\Big]$$

$$= \frac{1}{t}\Big[\ln(1+e^t) - \ln(1+1)\Big]$$

$$= \boxed{\frac{1}{t}\Big[\ln(1+e^t) - \ln(2)\Big]} \quad \forall t > 0$$