Chapter 6:

Ex 6.3. An experiment can result in any of the outcomes 1, 2 or 3.

(a) If there are two different wagers, with

$$r_1(1) = 4,\ r_1(2) = 8,\ r_1(3) = -10$$
$$r_2(1) = 6,\ r_2(2) = 12,\ r_2(3) = -16$$

is an arbitrage possible?

There is no arbitrage if we can solve the system:

$$\sum_{j=1}^{3} P_j r_i(j) = 0, \forall i : i=1,2 \quad \text{AND} \quad \sum_{i=1}^{3} P_i = 1, \quad P_i \geq 0 \ \forall i$$

$$\Rightarrow \begin{cases} P_1 r_1(1) + P_2 r_1(2) + P_3 r_1(3) = 0 \\ P_1 r_2(1) + P_2 r_2(2) + P_3 r_2(3) = 0 \\ P_1 + P_2 + P_3 = 1 \end{cases} \Rightarrow \begin{cases} 4P_1 + 8P_2 - 10P_3 = 0 & (1) \\ 6P_1 + 12P_2 - 16P_3 = 0 & (2) \\ P_1 + P_2 + P_3 = 1 & (3) \end{cases}$$

From (3) $\Rightarrow P_1 = 1 - P_2 - P_3$. Replace this in (1) & (2) to get:

$$\Rightarrow \begin{cases} 4 - 4P_2 - 4P_3 + 8P_2 - 10P_3 = 0 \\ 6 - 6P_2 - 6P_3 + 12P_2 - 16P_3 = 0 \end{cases} \Rightarrow \begin{cases} 4 + 4P_2 - 14P_3 = 0 \quad (4) \\ 6 + 6P_2 - 22P_3 = 0 \overset{\div 6}{\Rightarrow} 1 + P_2 - \frac{22}{6}P_3 = 0 \end{cases}$$

Hence, $P_2 = \frac{22}{6}P_3 - 1$. Replace this in (4) to get:

$$4 + \frac{88}{6}P_3 - 4 - 14P_3 = 0 \Rightarrow P_3\left(\frac{88}{6} - 14\right) = 0 \Rightarrow \boxed{P_3 = 0}$$

We can rewrite the original system as:

$$\begin{cases} 4P_1 + 8P_2 = 0 \\ 6P_1 + 12P_2 = 0 \\ P_1 + P_2 = 1 \end{cases} \Rightarrow \begin{cases} 4 - 4P_2 + 8P_2 = 0 \Rightarrow 4 + 4P_2 = 0 \Rightarrow \boxed{P_2 = -1} \\ P_1 = 1 - P_2 \Rightarrow \boxed{P_1 = 2} \end{cases}$$

Therefore, the solution $\vec{P} = (2, -1, 0)$ is not a valid probability vector. It follows that, by the arbitrage Theorem, that there is a possible arbitrage. (Try $\vec{x} = (x_1 = 2, x_2 = -1)$)

(b) If there are three different wagers, with

$$r_1(1) = 6,\ r_1(2) = -3,\ r_1(3) = 0$$
$$r_2(1) = -2,\ r_2(2) = 0,\ r_2(3) = 6$$
$$r_3(1) = 10,\ r_3(2) = 10,\ r_3(3) = x$$

what must x equal if there is no arbitrage?

For no arbitrage we need, for each i:

$$\sum_{j=1}^{3} P_j r_i(j) = 0 \quad \text{and} \quad \sum_{i=1}^{3} P_i = 1, \quad P_i \geq 0 \ \forall i$$

which means:

$$\begin{cases} P_1 r_1(1) + P_2 r_1(2) + P_3 r_1(3) = 0 \\ P_1 r_2(1) + P_2 r_2(2) + P_3 r_2(3) = 0 \\ P_1 r_3(1) + P_2 r_3(2) + P_3 r_3(3) = 0 \\ P_1 + P_2 + P_3 = 1 \end{cases} \Rightarrow \begin{cases} 6P_1 - 3P_2 = 0 \\ -2P_1 + 6P_3 = 0 \\ 10P_1 + 10P_2 + xP_3 = 0 \\ P_1 + P_2 + P_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 2P_1 - P_2 = 0 \\ P_1 - 3P_3 = 0 \\ 10P_1 + 10P_2 + xP_3 = 0 \ (*) \\ P_1 + P_2 + P_3 = 1 \end{cases} \Rightarrow \begin{cases} P_2 = 2P_1 \\ P_3 = \frac{1}{3}P_1 \end{cases} \Rightarrow P_1 + 2P_1 + \frac{1}{3}P_1 = 1 \Rightarrow \boxed{P_1 = \frac{3}{10}} \ \boxed{P_2 = \frac{6}{10}} \ \boxed{P_3 = \frac{1}{10}}$$

From which it follows: $\overset{(*)}{10\left(\frac{3}{10}\right) + 10\left(\frac{6}{10}\right) + x\left(\frac{1}{10}\right) = 0}$

$$\Rightarrow 3 + 6 + \frac{x}{10} = 0 \Rightarrow -9 = \frac{x}{10} \Rightarrow \boxed{x = -90}$$

Therefore, $x = -90$ if there is no arbitrage.

EX 6.6: Initial price of the stock is 100. After one period is assumed to be either 200 or 50. One has the option of purchasing a put option with strike price 150 after one period. To determine the value of P - the cost of one put option - if there is no arbitage, we consider two different wagers:

(1) sell the stock and (2) sell the option

(1) $$\text{P.V. return} = \begin{cases} -200(1+r)^{-1} + 100 & \text{if } S(1) = 200 \\ -50(1+r)^{-1} + 100 & \text{if } S(1) = 50. \end{cases}$$

Let p be the probabity that $S(1) = 200$, so $(1-p)$ prob. $S(1) = 50$. then

$$E[\text{return}] = p[-200(1+r)^{-1} + 100] + (1-p)[-50(1+r)^{-1} + 100]$$

$$0 = -200p(1+r)^{-1} + \cancel{100p} - 50(1+r)^{-1} + 100 + 50p(1+r)^{-1} - \cancel{100p}$$

$$\Rightarrow \quad p = \frac{1+2r}{3} \quad \Rightarrow \quad 1-p = \frac{2-2r}{3}$$

(2) $$\text{P.V return} = \begin{cases} -P & \text{if } S(1) = 200 \\ 100(1+r)^{-1} - P & \text{if } S(1) = 50 \end{cases}$$

then

$$E[\text{return}] = -pP + (1-p)[100(1+r)^{-1} - P]$$

$$= -\cancel{pP} + 100(1+r)^{-1} - P - 100p(1+r)^{-1} + \cancel{pP}$$

$$= 100(1+r)^{-1}[1-p] - P$$

$$= 100(1+r)^{-1}\left[\frac{2-2r}{3}\right] - P = 0$$

$$\Rightarrow \quad P = \frac{100(2-2r)}{3(1+r)} \quad \Rightarrow \quad \boxed{P = \frac{200(1-r)}{3(1+r)}}$$

//

We can see that the call and put prices satisfy the put-call option parity formula since:

$$S+P-C = 100 + \frac{200(1-r)}{3(1+r)} - \frac{50+100r}{3(1+r)}$$

$$= \frac{300(1+r)+200-200r-50-100r}{3(1+r)}$$

$$= \frac{300+300r-300r+150}{3(1+r)}$$

$$= \frac{450}{3(1+r)} = 150(1+r)^{-1} = K(1+r)^{-1}$$

Ex 6.12: the up probability is given by:

$$p = \frac{1+r-d}{u-d} = .7380, \quad \text{since } u=\frac{11}{10},\ d=\frac{10}{11},\ r=0.05.$$

the bet will pay off if at least 2 of the first 3 moves are up. Assumming no arbitrage, we get:

$$\boxed{C = (1.05)^{-3}\,100\left((.7380)^3 + 3(.7380)^2(.2620)\right)}$$

Chapter 7: (unit of time = 1 year)

EX 7.2: The prices of a security follow a geometric B.M with $\mu = .12$ and $\sigma = .24$. Suppose $S(0) = 40$.

What is the probability that a call option having four months until expiration and with strike price $K = 42$, will be exercised?

Sol: That a call option will be exercised means that the strike price K is less than the price at the moment of exercising it.

$$P(\text{call exercised}) = P(S(T) > K)$$

Consider $S(T) = S(4/12) = S(1/3)$ since unit is 1 year

$$P\left(S(1/3) > 42\right) = P\left(\frac{S(1/3)}{S(0)} > \frac{42}{S(0)}\right) \quad ; \text{ since } S(0) > 0$$

$$= P\left(\log\left[\frac{S(1/3)}{S(0)}\right] > \log\left(\frac{42}{40}\right)\right) \quad ; \text{ since log is an increasing function}$$

$$= P(X > \log 1.05), \quad \ldots \text{ by arithmetic.}$$

where $X \sim \text{Normal}\left(\frac{.12}{3}, \frac{.24}{\sqrt{3}}\right) = \text{Normal}\left(.04, \frac{.24}{\sqrt{3}}\right)$.

Normalizing X we can find this probability:

$$P(X > 1.05) = P\left(\frac{X - .04}{.24/\sqrt{3}} > \frac{\ln(1.05) - .04}{.24/\sqrt{3}}\right)$$

$$= P(X_{0,1} > 0.06343755)$$

$$= 1 - \Phi(0.06343755)$$

$$\approx \boxed{0.4747}$$

Hence, there is close to 47% chance the option will be exercised.

EX. 7.3: If the interest rate is 8%, what is the risk-neutral valuation of the call option specified in Exercise 7.2?

Sol: Using the Black-Scholes option pricing formula with parameters:

$S(0) = 40$, $K = 42$, $t = 4/12 = 1/3$, $r = 0.08$, $\sigma = .24$, we get:

$$w = \frac{rt + \sigma^2 t/2 - \log(K/S(0))}{\sigma\sqrt{t}} = \frac{0.08 \times \frac{1}{3} + (.24)^2 \times \frac{1}{6} - \log(42/40)}{.24/\sqrt{3}}$$

$$= \frac{-0.0125235}{0.138564}$$

$$= -0.0903806.$$

We compute the cost C:

$$C = S(0)\Phi(w) - Ke^{-rt}\Phi(w - \sigma\sqrt{t})$$

$$= 40\,\Phi(w) - 42e^{\frac{-0.08}{3}}\Phi(w - .24/\sqrt{3})$$

$$\approx \boxed{1.8137}$$

\#8. Consider a stock with $S(0) = 50 \begin{array}{l} S(1) = 80 \\ S(1) = 30 \end{array}$. In the language of the arbitrage Theorem, there are $n = 1$ bets and $m = 2$ possible states, and the profit return matrix is $R = r_{ij} = (30, -20)$.

Here we are in the no arbitrage case since: (by arbitrage theorem)

$\sum_{j=1}^{m} P_j r_i(j) = 0$, $\forall i = 1$, $\sum_{j=1}^{n} P_j = 1 \Rightarrow \sum_{j=1}^{2} P_j r_1(j) = 0$ AND $P_1 + P_2 = 1$

$\Rightarrow P_1 30 + P_2(-20) = 0$ AND $P_1 + P_2 = 1 \Rightarrow P_1 = 1 - P_2$

$\Rightarrow (1 - P_2)30 - 20P_2 = 30 - 30P_2 - 20P_2 = 0 \Rightarrow 30 = 50P_2 \Rightarrow \boxed{P_2 = \frac{3}{5}} \Rightarrow \boxed{P_1 = \frac{2}{5}}$

There exists a probability vector $(P_1, P_2) = (\frac{2}{5}, \frac{3}{5})$ that gives a neutral risk explanation for the price of the stock.

#9. Suppose $S(0) = 50$ and $S(1) = \begin{cases} 80; \\ 50; \\ 30 \end{cases}$. In the setting of the arbitrage theorem we have $n=1$, $m=3$ and the profit return matrix $R = (30, 0, -20)$

(a) Find a single vector (P_1, P_2, P_3) that explains the stock under no-arbitrage.

We want to solve: $\sum_{j=1}^{3} P_j r_1(j) = 0$ AND $\sum_{j=1}^{3} P_j = 1$, $P_j \geq 0 \; \forall j$.

$$\Rightarrow \begin{cases} 30P_1 + 0P_2 - 20P_3 = 0 \\ P_1 + P_2 + P_3 = 1 \end{cases} \Rightarrow \begin{cases} 20P_3 = 30P_1 \\ P_2 = 1 - P_1 - P_3 \end{cases} \Rightarrow \begin{cases} P_3 = \frac{3}{2}P_1 \\ P_2 = 1 - P_1 - \frac{3}{2}P_1 \Rightarrow P_2 = 1 - \frac{5}{2}P_1 \end{cases}$$

There are infinitely many solutions parametrized by P_1 as follows:

$(P_1, P_2, P_3) = (P_1, 1 - \frac{5}{2}P_1, \frac{3}{2}P_1)$. Not all of them will work since we need to make sure $1 - \frac{5}{2}P_1 \geq 0$ and $\frac{3}{2}P_1 \geq 0$.

One possible choice is $\boxed{P_1 = \frac{1}{30}} \Rightarrow P_2 = 1 - \frac{5}{2}\cdot\frac{1}{30} = \boxed{\frac{55}{60} = P_2}$, $P_3 = \frac{3}{2}\frac{1}{30} = \boxed{\frac{1}{20} = P_3}$

Thus, $\boxed{(P_1, P_2, P_3) = (\frac{2}{60}, \frac{55}{60}, \frac{3}{60})}$

(b) The call option with strike $K = 60$ at time 1 has price C. If you rely on the probability vector in part (a), what is C?

$$\text{P.V. return on option} = \begin{cases} -C & \text{if } S(1) = 80 \\ 10(1.1)^{-1} - C & \text{if } S(1) = 50 \\ 30(1.1)^{-1} - C & \text{if } S(1) = 30 \end{cases}$$

$$E\left[\substack{\text{P.V. return} \\ \text{on option}}\right] = P_1(-C) + P_2\left[10(1.1)^{-1} - C\right] + P_3\left[30(1.1)^{-1} - C\right]$$

$= -C(P_1 + P_2 + P_3) + (1.1)^{-1}[10P_2 + 30P_3]$ ...... since $P_1 + P_2 + P_3 = 1$.

$= -C + (1.1)^{-1}[10P_2 + 30P_3]$

We want $E\left[\substack{\text{P.V. return} \\ \text{on option}}\right] = 0$, so that there is no arbitrage. It follows:

$C = (1.1)^{-1}[10P_2 + 30P_3]$.

Using $P_2 = \frac{55}{60}$ AND $P_3 = \frac{3}{60} \Rightarrow C = (1.1)^{-1}\left[\frac{550 + 90}{60}\right] \cong \boxed{9.6969 = C}$

(c) A different probability vector from the one in part (a) that also explains the no-arbitrage stock price would be $(\frac{2}{80}, \frac{75}{80}, \frac{3}{80})$, since, if:

$\boxed{P_1 = \frac{1}{40}} \Rightarrow P_2 = 1 - \frac{5}{2}\frac{1}{40} = \boxed{\frac{75}{80} = P_2}$ AND $P_3 = \frac{3}{2}\frac{1}{40} = \boxed{\frac{3}{80} = P_3}$

(d) If we rely on the vector in part (c), then the value of C is

$$C = (1.1)^{-1}\left[10 \cdot \frac{75}{80} + 30 \cdot \frac{3}{80}\right] = (1.1)^{-1}\left[\frac{750+90}{80}\right] \approx \boxed{9.5454 = C}$$

(e) Does that mean that any value of C at all can be justified using this stock price model? If not, what are the largest and smallest values of C that you can justify using the different possible no-risk probability vector?

Sol: We want to solve the following optimization problem:

$$\max(\min) \quad C(P_2, P_3) = (1.1)^{-1}[10P_2 + 30P_3]$$

$$\text{subject to} \quad P_2 = 1 - \tfrac{5}{2}P_1 \text{ AND } P_3 = \tfrac{3}{2}P_1 \text{ AND } 0 \le P_1 \le 1.$$

But this reduces to:

$$\max(\min) \quad C(P_1) = (1.1)^{-1}\left[10\left(1 - \tfrac{5}{2}P_1\right) + 30\left(\tfrac{3}{2}P_1\right)\right]$$
$$= (1.1)^{-1}[10 + 20P_1]$$

$$\text{subject to} \quad 0 \le P_1 \le 1.$$

Since this is a (increasing) linear function of P_1, the min/max values occurs at the endpoints $P_1 = 0$ AND $P_1 = 1$, respectively.

Therefore, the minimum value of C is $C = 10/1.1$ AND the maximum value of C is $C = 30/1.1$

7.1: If the volatility of a stock is $\sigma = .33$, find the standard deviation of

(a) $\log\left(\frac{S_d(n)}{S_d(n-1)}\right)$: since $S_d(n) = S\left(\frac{n}{365}\right)$, b/c $S_d(n)$ = price at the end of day n, $\Rightarrow$

$$\log\left(\frac{S_d(n)}{S_d(n-1)}\right) = \log\left(\frac{S(n/365)}{S((n-1)/365)}\right) \sim \text{Normal}\left(\mu\frac{1}{365}, \frac{1}{365}(.33)^2\right) \Rightarrow \boxed{\text{stdev} = \frac{.33}{\sqrt{365}}}$$

(b) $\log\left(\frac{S_m(n)}{S_m(n-1)}\right)$: since $S_m(n) = S\left(\frac{n}{12}\right)$, b/c $S_m(n)$ = price at the end of month n $\Rightarrow$

$$\log\left(\frac{S_m(n)}{S_m(n-1)}\right) = \log\left(\frac{S(n/12)}{S((n-1)/12)}\right) \sim \text{Normal}\left(\mu\frac{1}{12}, \frac{1}{12}(.33)^2\right) \Rightarrow \boxed{\text{stdev} = \frac{.33}{\sqrt{12}}}$$

6.10: Consider the model of section 6.2 with $n=1$.
To recreate the option by a combination of borrowing and buying the security, we want to find the combination that leads to the same payoff:

Let $S(0)=s$. Suppose $us > K > ds$. (u is the up factor and d is the down factor). Let y be the number of shares of the security we buy by borrowing x. Then, the return at time 1 is:

$$\underset{@\,t=1}{\text{return}^{(\text{shares})}} = \begin{cases} \overbrace{yus}^{\text{value}} - \overbrace{(1+r)x}^{\text{cost}}, & \text{if } S(1)=us \\ yds-(1+r)x, & \text{if } S(1)=ds \end{cases}$$

We want this return to replicate the return of a call option, (strike K)
The return on the call is

$$\underset{@\,t=1}{\text{return}^{(\text{option})}} = \begin{cases} us-K & \text{if } \overset{us}{\overset{\shortparallel}{S(1)}} > K \Rightarrow S(1)=us \\ 0 & \text{if } S(1)\leq K \Rightarrow S(1)=ds \end{cases}$$

Therefore:
$$\begin{cases} yus-(1+r)x = us-K & (*) \\ yds-(1+r)x = 0 & (**) \end{cases}$$
(from which we can solve for y and x)

$yds-(1+r)x=0 \Rightarrow yds=(1+r)x$. Replace in $(*)$:

$$yus-yds=us-K \Rightarrow \boxed{y=\frac{us-K}{us-ds}} //$$

Replace in $(**)$:

$$\left(\frac{us-K}{us-ds}\right)\cdot ds-(1+r)x=0 \Rightarrow \boxed{x=\frac{(us-K)ds}{(us-ds)(1+r)}} //$$