Chapter 8:

Ex. 8.1: Does the put-call option parity formula for European call and put options remain valid when the security pays dividends?

Sol: Yes, since the strike price is fixed at the end.

Ex. 8.2: For the model of Section 8.2.1, under the risk-neutral probabilities, what process does the security's price over time follow?

Sol: As stated in Section 8.2.1, under the risk-neutral probabilities, we have that $S(t) = S(0)e^{-ft}e^{W(t)}$, where $W \sim \text{Normal}\left(\left(r-\frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$

But, $S(t) = S(0)e^{-ft}e^{W(t)} = S(0)e^{W(t)-ft}$, so $W(t)-ft \sim \text{Normal}\left(\left(r-\frac{\sigma^2}{2}\right)t - ft, \sigma^2 t\right)$

$\iff W(t)-ft \sim \text{Normal}\left(\left(r-\frac{\sigma^2}{2}-f\right)t, \sigma^2 t\right)$.

This means that the security's price follow G.B.M with drift $r-\frac{\sigma^2}{2}-f$ and volatility σ.

Ex. 8.3: Find the no-arbitrage cost of a European (K,t) call option on a security that, at times t_{d_i} $(i=1,2)$, pays $f\,S(t_{d_i})$ as dividends, where $t_{d_1} < t_{d_2} < t$.

Sol: Following the model 8.2.2, we have that, starting with a single share at time 0, the market value of our portfolio at time y, call it $M(t)$, is:

$$M(t) = \begin{cases} S(t) & \text{if } t < t_{d_1} \\ \frac{1}{1-f}S(t) & \text{if } t_{d_1} \le t < t_{d_2} \\ \frac{1}{(1-f)^2}S(t) & \text{if } t > t_{d_2} \end{cases} \Rightarrow \text{since we reinvest immediately}$$

So, for $t > t_{d_2}$, we have: $S(t)/S(0) = (1-f)^2 M(t)/M(0) = (1-f)^2 e^{W(t)}$, since we assume $M(y)$ $(y \ge 0)$ to follow G.B.M with vol. σ and drift $r-\frac{\sigma^2}{2}$ (risk-neutral prob.). Therefore, the no-arbitrage cost is

$$C = e^{-rt}E[\text{return @ } t] = e^{-rt}E\left[(S(t)-K)^+\right] = e^{-rt}\left[\left(S(0)(1-f)^2e^{W(t)} - K\right)^+\right] = \underbrace{\boxed{C(S(0)(1-f)^2, K, t, r, \sigma)}}_{\text{Black-Scholes}}$$

Ex 8.5: Consider a European (K,t) call option whose return at expiration time is capped by the amount B. That is, the payoff at t is:

$$\min((S(t)-K)^+, B).$$

Explain how you can use the Black-Scholes formula to find the no-arbitrage cost of this option.

Sol: We will consider two investments I_1 and I_2 such that the payoff from I_1 minus the payoff from I_2 is equal to the payoff from the capped option. It will follow, by the law of one price, that the cost of the capped option must be equal to the difference of the cost of I_1 and I_2 for there not to be arbitrage.

I_1 = one (K,t) call option (on the same security as that of capped option).

I_2 = one $(K+B,t)$ call option (" " " ").

Consider the payoffs:

$$\text{Payoff of } I_1 \ (K,t)\text{-call option} = \begin{cases} S(t)-K & \text{if } S(t)>K \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Payoff of } I_2 \ (K+B,t)\text{-Call option} = \begin{cases} S(t)-(K+B) & \text{if } S(t)>K+B \\ 0 & \text{otherwise } (S(t)<K+B) \end{cases}$$

$$\text{Payoff of Capped option} = \begin{cases} S(t)-K \ ① & \text{if } \begin{cases} S(t)>K \text{ AND} \\ S(t)-K<B \end{cases} \\ B \ ② & \text{if } B< S(t)-K \\ 0 \ ③ & \text{if } S(t)<K \end{cases}$$

note, we assume $B\geq 0$.

Now, consider the difference of payoff of I_1 and I_2:

$$\text{payoff of } I_1 - I_2 = \begin{cases} \overset{I_1}{S(t)-K} - \overset{I_2}{0} = S(t)-K \ ① & \text{if } \begin{cases} S(t)>K \text{ AND} \\ S(t)-K<B \end{cases} \Leftrightarrow S(t)<K+B \\ S(t)-K-[S(t)-(K+B)] = B \ ② & \text{if } \begin{cases} S(t)>K \text{ AND} \\ S(t)-K>B \end{cases} \Leftrightarrow S(t)>K+B \\ 0-0=0 \ ③ & \text{if } S(t)<K \Rightarrow S(t)<K+B \ (\text{since } B\geq 0) \end{cases}$$

This shows that payoff of $I_1 - I_2$ is same as payoff of capped option.

Cost of $I_1 \Rightarrow C(S(0),K,t,r,\sigma)$

Cost of $I_2 \Rightarrow C(S(0),K+B,t,r,\sigma)$

$C(S(0),K,t,r,\sigma) - C(S(0),K+B,t,r,\sigma)$ = cost of difference = cost of capped opt

Ex 8.14 : Derive an approximation to the risk-neutral price of an American put option having parameters:

$S = 10$, $t = .25$, $K = 10$, $\sigma = .3$, $r = .06$.

Sol : Choose $n = 5$. Then,

$u = e^{\sigma\sqrt{t/n}} = e^{.3\sqrt{.05}} = 1.0694$; $d = e^{-\sigma\sqrt{t/n}} = e^{-.3\sqrt{.05}} = 0.9351$

$$p = \frac{1 + rt/n - d}{u - d} = \frac{1 + .06 \times .25/5 - 0.9351}{1.0694 - 0.9351} = 0.5056 \Rightarrow 1 - p = 0.4944.$$

$e^{-rt/n} = e^{-.06 \times \frac{.25}{5}} = 0.997$. $\quad S(t_k, i) = u^i d^{k-i} S$

The possible prices of the security at time t_5 are: $S(t_5, i) = u^i d^{5-i} \cdot S$.

$S(t_5, 0) = 10 \cdot u^0 d^{5-0} = 10 d^5 = 7.150$
$S(t_5, 1) = 10 \cdot u^1 d^{5-1} = 10 \cdot u \cdot d^4 = 8.177$
$S(t_5, 2) = 10 \cdot u^2 \cdot d^{5-2} = 10 \cdot u^2 \cdot d^3 = 9.351$
$S(t_5, 3) = 10 \cdot u^3 \cdot d^{5-3} = 10 \cdot u^3 \cdot d^2 = 10.694$
$S(t_5, 4) = 10 \cdot u^4 \cdot d^{5-4} = 10 \cdot u^4 \cdot d = 12.230$
$S(t_5, 5) = 10 \cdot u^5 \cdot d^{5-5} = 10 \cdot u^5 = 13.986$

Hence,

$V_5(i) = \max(K - u^i d^{n-i} \cdot S, 0) = \max(10 - u^i d^{n-i} \cdot 10, 0)$.

$V_5(0) = \max(10 - 7.150, 0) = 2.85$
$V_5(1) = \max(10 - 8.177, 0) = 1.823$
$V_5(2) = \max(10 - 9.351, 0) = 0.649$
$V_5(3) = \max(10 - 10.694, 0) = 0$
$V_5(4) = \max(10 - 12.230, 0) = 0$
$V_5(5) = \max(10 - 13.986, 0) = 0$

The possible prices of the security at time t_4 are: $S(t_4, i) = u^i d^{4-i} \cdot S$

$S(t_4, 0) = 10 \cdot u^0 \cdot d^{4-0} = 10 d^4 = 7.646$
$S(t_4, 1) = 10 \cdot u^1 \cdot d^{4-1} = 10 \cdot u \cdot d^3 = 8.744$
$S(t_4, 2) = 10 \cdot u^2 \cdot d^{4-2} = 10 \cdot u^2 \cdot d^2 = 10$
$S(t_4, 3) = 10 \cdot u^3 \cdot d^{4-3} = 10 \cdot u^3 \cdot d = 11.436$
$S(t_4, 4) = 10 \cdot u^4 \cdot d^{4-4} = 10 \cdot u^4 = 13.079$

$V_4(i) = \max(K - u^i d^{k-i} S,\ e^{-rT/n}[p V_{k+1}(i+1) + (1-p) V_{k+1}(i)])$

$V_4(i) = \max(10 - u^i d^{4-i} \cdot 10,\ 0.997[0.5056\, V_5(i+1) + 0.4944\, V_5(i)])$, for $i = 0,1,2,3,4$

$V_4(0) = \max(10 - 7.646,\ 0.997[0.5056(1.823) + 0.4944(2.85)])$
$= \max(2.354,\ 2.324) = 2.354.$

$V_4(1) = \max(10 - 8.744,\ 0.997[0.5056(0.649) + 0.4944(1.823)])$
$= \max(1.256,\ 1.226) = 1.256.$

$V_4(2) = \max(10 - 10,\ 0.997[0.5056(0) + 0.4944(0.649)])$
$= \max(0,\ 0.320) = 0.320.$

$V_4(3) = \max(10 - 11.436,\ 0.997[0.5056(0) + 0.4944(0)])$
$= \max(\text{negative \#},\ 0) = 0.$

$V_4(4) = \max(10 - 13.079,\ 0.997[0.5056(0) + 0.4944(0)])$
$= \max(\text{negative \#},\ 0) = 0$

the possible prices of the security at time t_3 are: $S(t_3, i) = u^i d^{3-i} \cdot S$

$S(t_3, 0) = 10 \cdot u^0 \cdot d^{3-0} = 10 \cdot d^3 = 8.177$
$S(t_3, 1) = 10 \cdot u^1 \cdot d^{3-1} = 10 \cdot u^1 d^2 = 9.351$
$S(t_3, 2) = 10 \cdot u^2 \cdot d^{3-2} = 10 \cdot u^2 \cdot d = 10.694$
$S(t_4, 3) = 10 \cdot u^3 \cdot d^{3-3} = 10 \cdot u^3 = 12.230$

$V_3(i) = \max(10 - u^i d^{3-i} \cdot 10,\ 0.997[0.5056\, V_4(i+1) + 0.4944\, V_4(i)])$, for $i = 0,1,2,3$

$V_3(0) = \max(10 - 8.177,\ 0.997[0.5056(1.256) + 0.4944(2.354)])$
$= \max(1.823,\ 1.793) = 1.823$

$V_3(1) = \max(10 - 9.351,\ 0.997[0.5056(0.320) + 0.4944(1.256)])$
$= \max(0.649,\ 0.780) = 0.780$

$V_3(2) = \max(10 - 10.694,\ 0.997[0.5056(0) + 0.4944(0.320)])$
$= \max(\text{negative \#},\ 0.158) = 0.158$

$V_3(3) = \max(10 - 12.230,\ 0.997[0.5056(0) + 0.4944(0)])$
$= \max(\text{negative \#},\ 0) = 0$

The possible prices of the security at time t_2 are : $S(t_2,i) = u^i d^{2-i} \cdot s$, $i=0,1,2$

$S(t_2,0) = 10 \cdot u^0 \cdot d^{2-0} = 10 \cdot d^2 = 8.744$

$S(t_2,1) = 10 \cdot u^1 d^{2-1} = 10 \cdot u d = 10$

$S(t_2,2) = 10 \cdot u^2 \cdot d^{2-2} = 10 \cdot u^2 = 11.436$.

$V_2(i) = \max(10 - u^i d^{2-i} \cdot 10, 0.997[0.5056\, V_3(i+1) + 0.4944\, V_3(i)])$, for $i=0,1,2$

$V_2(0) = \max(10 - 8.744, 0.997[0.5056(0.780) + 0.4944(1.823)])$

$= \max(1.256, 1.292) = 1.292$

$V_2(1) = \max(10 - 10, 0.997[0.5056(0.158) + 0.4944(0.780)])$

$= \max(0, 0.464) = 0.464$

$V_2(2) = \max(10 - 11.436, 0.997[0.5056(0) + 0.4944(0.158)])$

$= \max(\text{negative } \#, 0.078) = 0.078$

The possible prices of the security at time t_1 are : $S(t_1,i) = u^i d^{1-i} \cdot s$, $i=0,1$

$S(t_1,0) = 10 \cdot u^0 \cdot d^{1-0} = 10 \cdot d = 9.351$

$S(t_1,1) = 10 \cdot u^1 \cdot d^{1-1} = 10 \cdot u = 10.694$

$V_1(i) = \max(10 - u^i d^{1-i} \cdot s, 0.997[0.5056\, V_2(i+1) + 0.4944\, V_2(i)])$, $i=0,1$

$V_1(0) = \max(10 - 9.351, 0.997[0.5056(0.464) + 0.4944(1.292)])$

$= \max(0.649, 0.871) = 0.871$

$V_1(1) = \max(10 - 10.694, 0.997[0.5056(0.078) + 0.4944(0.464)])$

$= \max(\text{negative } \#, 0.268) = 0.268$

Now, we can compute the price at time 0:

$V_0(0) = \max(10 - 10, 0.997[0.5056(0.268) + 0.4944(0.871)])$

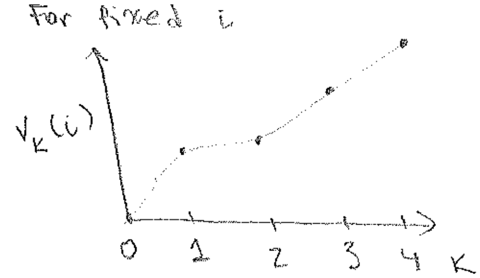
$= \max(0, 0.564) = 0.564$

That is, the risk-neutral price of the put option is approximately $\boxed{0.564}$ //

Ex 8.12 : Using the notation of Section 8.3, which of the following statements do you think are true?. Explain your reasoning.

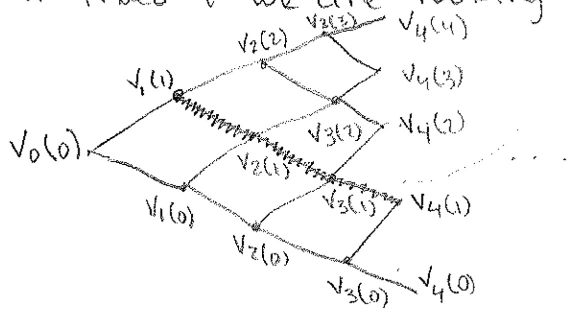
(a) $V_k(i)$ is nondecreasing in k for fixed i.

YES For fixed i



Recall, $V_k(i)$ = the expected return @ time t_k of the i^{th} branch.
(assuming $S(t_k) = u^i d^{k-i} S(0)$, and that we didn't previously exercise optimal strategy for exercising put).

For fixed i we are looking at a fixed path, e.g.:



$V_1(1), V_2(1), V_3(1), V_4(1), \ldots$

Since we are following a "down" path, i.e, a path where the price of the security goes down, we expect to be able to exercise our put and hence, the value of it must go up (or stay the same).

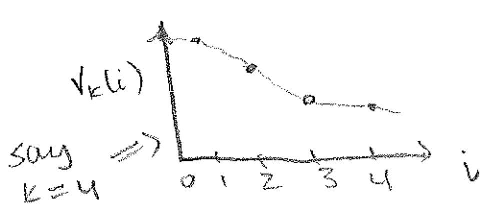
thus, $V_k(i)$ is nondecreasing in k for fixed i.

(b) $V_k(i)$ is non increasing in k for fixed i

NO : for same reasons as explained in (a).

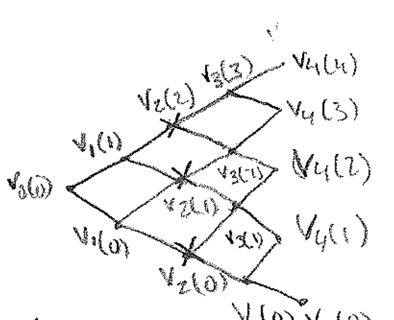
(c) $V_k(i)$ is non decreasing in i for fixed k.

NO For fixed k



we are looking at a cross section such as:
$V_2(0), V_2(1), V_2(2)$ or $V_3(0), V_3(1), V_3(2), V_3(3)$

In such a path, the value of $V_k(i) \leq V_k(i-1)$ (for fixed k), since lower values of i correspond to paths where the price of the security is lower, so we expected to exercise the put leading to a higher value of V_k.
For example: $V_2(0) \geq V_2(1) \geq V_2(2)$, since at $V_2(0)$, we are in a all down path whereas in $V_2(2)$ we are in a all up path (less likely to exercise put).



(d) $V_k(i)$ is non-increasing in i for fixed k ⇒ YES as explain before.