Chapter 8:

Ex. 8.6: the current price of a security is s. Consider an investment whose cost is s and whose payoff at time 1 is, for a specified choice of β satisfying $0 < \beta < e^r - 1$, given by:

$$\text{return} = \begin{cases} (1+\beta)s & \text{if } S(1) \leq (1+\beta)s, \\ (1+\beta)s + \alpha(S(1) - (1+\beta)s) & \text{if } S(1) \geq (1+\beta)s. \end{cases}$$

Determine the value of α if this investment (whose payoff is both uncapped and always greater than the initial cost of the investment) is not to give rise to an arbitrage.

Sol: Under the risk neutral Geometric Brownian motion, the expected return from this investment is:

$$\mathbb{E}[(1+\beta)s + \alpha(S(1) - (1+\beta)s)^+] = (1+\beta)s + \alpha e^r C(s, 1, (1+\beta)s, \sigma, r)$$

In order not to have arbitrage, and since the current price of a security is s and the cost of the investment is also s, the payoff of both security and investment should be the same, i.e., sell the security and invest s to get se^r @ time 1. Thus, the return from the investment should be equal to se^r.

$$(1+\beta)s + \alpha e^r C(s, 1, (1+\beta)s, \sigma, r) = se^r \Rightarrow$$

$$\alpha e^r C(s, 1, (1+\beta)s, \sigma, r) = s(e^r - (1+\beta)) \Rightarrow$$

$$\boxed{\alpha = \frac{s(e^r - (1+\beta))}{e^r C(s, 1, (1+\beta)s, \sigma, r)}}$$

Ex 8.7: the following investment is being offered on a security whose current price is S. For an initial cost of S and for the value β of your choice (provided that $0<\beta<e^r-1$), your return after one year is given by

$$\text{return} = \begin{cases} (1+\beta)S & \text{if } S(1) \leq (1+\beta)S, \\ S(1) & \text{if } (1+\beta)S \leq S(1) \leq K, \\ K & \text{if } S(1) > K, \end{cases}$$

where $S(1)$ is the price of the security at the end of one year. In other words, at the price of capping your maximum return at time 1 you are guaranteed that your return at time 1 is at least $1+\beta$ times your original payment.

Show that this investment (which can be bought or sold) does not give rise to an arbitrage when K is such that

$$C(S,1,K,\sigma,r) = C(S,1,S(1+\beta),\sigma,r) + S(1+\beta)e^{-r} - S,$$

where $C(S,t,K,\sigma,r)$ is the Black-Scholes formula.

Sol: Under the risk neutral Geometric Brownian motion, provided that $K>(1+\beta)S$, the expected return from this investment is:

$$\mathbb{E}\left[(1+\beta)S + (S(1)-(1+\beta)S)^+ - (S(1)-K)^+\right]$$
$$= (1+\beta)S + e^r C(S,1,(1+\beta)S,\sigma,r) - e^r C(S,1,K,\sigma,r)$$

Just like in Ex 8.6, there is no arbitrage if this pay off is equal to e^rS (selling the security and invest it for one time period).

In this case:

$$(1+\beta)S + e^r C(S,1,(1+\beta)S,\sigma,r) - e^r C(S,1,K,\sigma,r) = Se^r \Rightarrow$$
$$e^r C(S,1,K,\sigma,r) = (1+\beta)S - Se^r + e^r C(S,1,(1+\beta)S,\sigma,r) \Rightarrow$$
$$\boxed{C(S,1,K,\sigma,r) = (1+\beta)Se^{-r} - S + C(S,1,(1+\beta)S,\sigma,r)}$$

Note that $S(1+\beta)e^{-r} < S \Rightarrow S(1+\beta)e^{-r} - S < 0$ AND $C(S,1,K,\sigma,r)$ is decreasing in K. Therefore, $K > (1+\beta)S$

Ex 8.8: Show that, for $f < r$,

$$C(Se^{-ft}, t, K, \sigma, r) = e^{-ft} C(S, t, K, \sigma, r-f)$$

Sol: $C(Se^{-ft}, t, K, \sigma, r) = e^{-rt} \mathbb{E}[(Se^{-ft}e^{W} - K)^+]$,

where $W \sim \text{Normal}\left(\left(r - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$.

Let Z be standard Normal, i.e., $Z = \dfrac{W - (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$

then, $Z\sigma\sqrt{t} + (r - \frac{\sigma^2}{2})t = W$. Replacing into above equation:

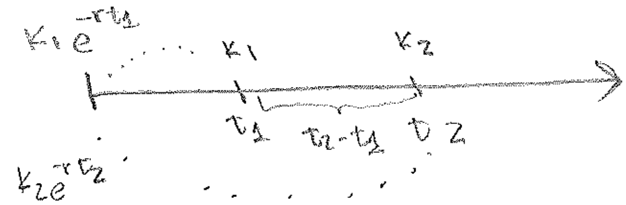
$$
\begin{aligned}
C(Se^{-ft}, t, K, \sigma, r) &= e^{-rt} \mathbb{E}[(Se^{-ft}e^{W} - K)^+] \\
&= e^{-rt} \mathbb{E}[(Se^{-ft}e^{Z\sigma\sqrt{t} + (r - \frac{\sigma^2}{2})t} - K)^+] \\
&= e^{-rt} \mathbb{E}[(Se^{Z\sigma\sqrt{t} + (r - f - \frac{\sigma^2}{2})t} - K)^+] \\
&= e^{-ft} e^{-(r-f)t} \mathbb{E}[(Se^{Z\sigma\sqrt{t} + (r - f - \frac{\sigma^2}{2})t} - K)^+] \\
&= e^{-ft} C(S, t, K, \sigma, r-f) \quad //
\end{aligned}
$$

EX 8.10. A (K_1, t_1, K_2, t_2) double call option is one that can be exercised either at time t_1 with strike price K_1 or at time t_2 $(t_2 > t_1)$ with strike price K_2.

(a) Argue that you would never exercise at time t_1 if

$$K_1 > e^{-r(t_2-t_1)} K_2.$$

Sol:



If you exercise at t_1, you pay the present value $K_1 e^{-rt_1}$

If you exercise at t_2, you pay the present value $K_2 e^{-rt_2}$

But, by assumption: $K_1 > e^{-r(t_2-t_1)} K_2 = e^{-rt_2 + rt_1} K_2 = e^{-rt_2} e^{rt_1} K_2$

$\Rightarrow K_1 e^{-rt_1} > K_2 e^{-rt_2}$, So you pay more if exercise at t_1.

therefore, you would never exercise at time t_1.

(b) Assume that $K_1 < e^{-r(t_2-t_1)} K_2$. Argue that there is a value x such that the option should be exercised at time t_1 if $S(t_1) > x$ and not exercised if $S(t_1) < x$.

Sol: If the option is not exercised at t_1, the risk-neutral expected return is $C(s_1, t_2 - t_1, K_2, \sigma, r)$, letting $S(t_1) = s_1$.

If the option is exercised at t_1, the value of it is : $S_1 - K_1$

Hence, one should exercise at time t_1 if:

$$S_1 - K_1 > C(S_1, t_2 - t_1, K_2, \sigma, r)$$

$$\Leftrightarrow \quad S_1 > K_1 + C(S_1, t_2 - t_1, K_2, \sigma, r).$$

So, the value of x is given by: $\boxed{x = K_1 + C(S_1, t_2 - t_1, K_2, \sigma, r)}$

Ex 8.15: An American asset-or-nothing call option (with parameters K, F and expiration time t) can be exercised any time up to t.
If the security's price when the option is exercised is K or higher, then the amount F is returned;
If the security's price when the option is exercised is less than K, then nothing is returned.
Explain how you can use the multiperiod binomial model to approximate the risk-neutral price of an American asset-or-nothing call option.

Sol: this option should be exercised whenever the price is at least K.
Note that it can be explicitly priced by using the formula in Chapter 3 for the maximum by time t of a Brownian motion.
It can be approximated by a N period binomial model:
Take the same states as used in pricing an American put option, and work backwards to obtain $V_0(0)$.
It takes less work than determining the risk neutral cost of an American put option because the optimal strategy for the asset-or-nothing is known in advance. (instead of using $K - u^i d^{k-i}s$, use F).

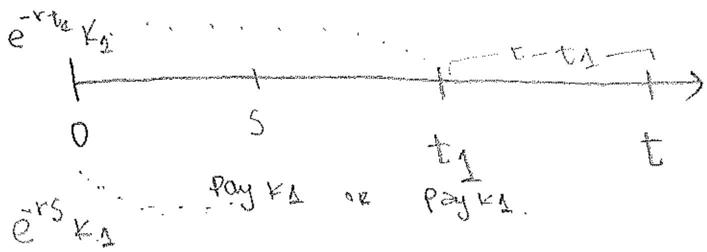
EX 8.16: Derive an approximation to the risk-neutral price of an American asset-or-nothing call option when:

$$s = 10, t = .25, K = 11, F = 20, \sigma = .3, r = .06.$$

Sol: instead of using $K - u^i d^{k-i} s$, use F and follow the same procedure as in previous homework.

Ex 8.9: An option on an option, sometimes called a compound option, is specified by the parameter pairs (K_1, t_1) and (K, t), where $t_1 < t$. The holder of such a compound option has the right to purchase, for the amount K_1, a (K, t) call option on a specified security. This option to purchase the (K, t) call option can be exercised any time up to time t_1.

(a) Argue that the option to purchase the (K,t) call option would never be exercised before its expiration time t_1.



Suppose you exercise the option at time S, such that $S < t_1$. Then you would have to pay K_1 @ time S, or P.V. $= e^{-rS} K_1$.
Suppose you exercise the option at time t_1.
Then you would have to pay K_1 @ time t_1, or P.V. $= e^{-rt_1} K_1$.
But $e^{-rS} K_1 > e^{-rt_1} K_1$, since $S < t_1$. Hence, earlier excersice always results in paying more. A dominating strategy is to exercise @ t_1.

(b) Argue that the option to purchase the (K,t) call option should be exercised if and only if $S(t_1) \geq x$, where x is the solution of

$$K_1 = C(x, t - t_1, K, \sigma, r),$$

$C(s, t, K, \sigma, r)$ is the Black-Scholes formula, and $S(t_1)$ is the price of the security at time t_1.

Sol: ($\Rightarrow$) Suppose $S(t_1) \geq x$, where x is the solution of $K_1 = C(x, t-t_1, K, \sigma, r)$.
Note that $C(x, t-t_1, K, \sigma, r)$ is the value of the call at time t_1 when $S(t_1) = x$. So, if $S(t_1) \geq x$, the price of the call will be at most K_1, so that buying the call yields a positive balance.

(c) Argue that there is a unique value of x that satisfies the preceding identity.

Sol: this follows because $C(y, t-t_1, K, \sigma, r)$ is a strictly increasing function of y.

(d) Argue that the unique no-arbitrage cost of this compound option can be expressed as

$$\text{no-arbitrage cost of compound option} = e^{-rt_1} \mathbb{E}[C(se^W, t-t_1, K, \sigma, r) I(se^W > x)]$$

Sol: this follows because the optimal policy is to exercise the option to purchase the call option at time t_1 if and only if $S(t_1) \geq x$.