M451 - Enrique Areyan - Spring 2015 - HW 8

$u(x) = 1-e^{-x}$

Chapter 9, Ex 1:

(1) $X \sim f_1(x) = e^{-x}, \quad x > 0$

(2) $Y \sim f_2(x) = 1/2, \quad 0 < x < 2.$

He should choose $\max\{E[u(X)], E[u(Y)]\}$.

(1) $E[u(X)] = \int_0^\infty (1-e^{-x})e^{-x}\,dx = \int_0^\infty e^{-x} - e^{-2x}\,dx$

$= \int_0^\infty e^{-x}dx - \int_0^\infty e^{-2x}\,dx = \left[-e^{-x}\right]_0^\infty - \left[-\frac{e^{-2x}}{2}\right]_0^\infty$

$= \left(\lim_{t\to\infty} -e^{-t}\right) + e^0 - \frac{1}{2}\left[\left(\lim_{t\to\infty} -e^{-2t}\right) + e^0\right]$

$= 0 + 1 - \frac{1}{2}[0+1] = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$

(2) $E[u(Y)] = \int_0^2 (1-e^{-x})\frac{1}{2}\,dx = \frac{1}{2}\left[\int_0^2 1\,dx - \int_0^2 e^{-x}\,dx\right]$

$= \frac{1}{2}\left[[x]_0^2 - [-e^{-x}]_0^2\right] = \frac{1}{2}\left[2 + (e^{-2} - e^0)\right]$

$= \frac{1}{2}[1+e^{-2}] \approx \boxed{0.56767}$

So $E[u(Y)] > E[u(X)]$, choose investment 2.

Ex 2:

X	$P(X=x)$	$x \cdot P(X=x)$
-1	$\frac{4}{10}$	$\frac{-4}{10} = \frac{-40}{100}$
$\frac{2}{10}$	$\frac{5}{10}$	$\frac{10}{100}$
$\frac{25}{10}$	$\frac{1}{10}$	$\frac{25}{100}$

total: $-\frac{40}{100} + \frac{10}{100} + \frac{25}{100}$

$= \frac{-5}{100} < 0$

This shows that this investment has a negative expected value:

$E[X] = \frac{-5}{100} < 0.$

A risk-averse individual would invest nothing on it.

So, the optimal value is $\boxed{a=0}$

Ex 3: In example 9.2a it was shown that the optimal value of α (the fraction of wealth to invest) is:

$\alpha = 2p - 1$.

Now, if $p \leq \frac{1}{2}$, then $\alpha = 2p - 1 \leq 0 \Rightarrow \alpha \leq 0$.

Since α is a fraction of the investment it follows $\alpha \geq 0$.

together $\alpha \leq 0$ and $\alpha \geq 0$ imply $\boxed{\alpha = 0}$

Ex 5: $r_1 = .15$, $V_1 = .20$; $r_2 = .18$, $V_2 = .25$

$u(x) = 1 - e^{-0.005x}$ AND $\rho = 0$

Since $\rho = \rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = 0 \Rightarrow Cov(X,Y) = 0$, so $C(1,2) = 0$.

By example 9.3b: $w_1 = y$, $w_2 = 100 - y$.

$E[W] = 118 - .03y$.

$Var[W] = \sum_{i=1}^{2} w_i^2 V_i^2 + \sum_{i=1}^{2} \sum_{i \neq j} w_i w_j C(i,j)$, where $C(i,j) = Cov(R_i, R_j) \Rightarrow$

$Var(W) = (.20y)^2 + (.25)^2(100-y)^2 + 2w_1w_2 \overset{0}{C(1,2)} \Rightarrow$

$Var(W) = 0.1025y^2 - 12.5y + 625$

we want to maximize: $E[W] - \frac{b}{2}Var(W) = 118 - 0.03y - 0.0025[Var(W)]$

$\Rightarrow f(y) = 118 - 0.03y - 0.0025[0.1025y^2 - 12.5y + 625]$

$\frac{df}{dy} = -0.03 - 0.0025[2 \times 0.1025y - 12.5] = -0.03 - 0.0005125y + 0.03125$
$= 0.00125 - 0.0005125y = 0$

$\Rightarrow y = \frac{0.00125}{0.0005125} = \frac{12500}{5125} = \boxed{\frac{100}{41} = y}$

The optimal portfolio is where $y = \frac{100}{41}$, i.e.,

$\boxed{w_1 = y = \frac{100}{41}}$ AND $w_2 = 100 - y = 100 - \frac{100}{41} = \boxed{\frac{4{,}000}{41} = w_2}$

Ex 7: Show that the percentage of one's wealth that should be invested in each security when attempting to maximize $E[\log(W)]$ does not depend on the amount of initial wealth.

Sol: Let W_n denote the wealth after the nth investment.
Let W_0 denote the initial wealth.
Let α_i be the percentage to invest in investment i
And X_i be the R.V. corresponding to investment i.

then, After one period we have $W_1 = \alpha_1 X_1 W_0$. After n periods:

$$W_n = \alpha_n X_n \alpha_{n-1} X_{n-1} \cdots \alpha_1 X_1 W_0.$$ Apply log to both sides:

$$\log(W_n) = \log(\alpha_n X_n \alpha_{n-1} X_{n-1} \cdots \alpha_1 X_1 W_0) = \log(W_0) + \sum_{i=1}^{n} \log(\alpha_i X_i)$$

thus,

$$E[\log(W_n)] = E\left[\log(W_0) + \sum_{i=1}^{n} \log(\alpha_i X_i)\right] \iff$$

$$E[\log(W_n)] = \log(W_0) + \sum_{i=1}^{n} E[\log(\alpha_i X_i)]$$, since E is linear and W_0 is a constant.

When optimizing this quantity $\log(W_0)$ will vanish as soon as we take the derivative. therefore, the optimal α_i's do not depend on W_0.

Ex 9: Does the percentage of one's wealth to be invested in each security when attempting to maximize the approximation (9.5) depend on initial wealth when $U(x) = \log(x)$?

Sol:

$$(9.5): \quad E[U(W)] \approx U(E[W]) + U''(E[W])\, Var(W)/2,$$

where $W = W_0 \sum_{i=1}^{n} \alpha_i X_i \Rightarrow E[W] = W_0 E\left[\sum_{i=1}^{n} \alpha_i X_i\right]$. Also, $U(x) = \log(x)$.

So, $U'(x) = \frac{1}{x} \Rightarrow U''(x) = \frac{-1}{x^2}$. Replacing this:

Maximize: $E[U(W)]$ is approximately equal to maximize:

$$\log(E[W]) - \frac{Var(W)}{2 \cdot E[W]^2} = \log\left(W_0 E\left[\sum_{i=1}^{n} \alpha_i X_i\right]\right) - \frac{Var(W)}{2 \cdot W_0^2 E\left[\sum_{i=1}^{n} \alpha_i X_i\right]^2}$$

where, $Var(W) = Var(w_0 \sum_{i=1}^{n} \alpha_i X_i) = W_0^2\, Var(\sum_{i=1}^{n} \alpha_i X_i)$. thus,
we wish to maximize:

$$\log(w_0) + \log(E[\sum_{i=1}^{n}\alpha_i X_i]) - \frac{W_0^2\, Var(\sum_{i=1}^{n}\alpha_i X_i)}{2W_0^2\, E[\sum_{i=1}^{n}\alpha_i X_i]},$$

which does <u>not</u> depend on initial wealth W_0 in optimizing for α_i.

<u>8</u> (not from book).

Let $P = xR_x + yR_y + zR_z$

a) If $(x,y,z) = (300,0,0)$, what are $E[P]$ and $Var(P)$?

i) $E[P] = E[xR_x + yR_y + zR_z] = E[300R_x + 0R_y + 0R_z] = 300\, E[R_x]$
$= 300 \cdot 0.11 = \boxed{33}$

(ii) $Var(P) = Var(xR_x + yR_y + zR_z) = Var(300R_x + 0R_y + 0R_z) = 300^2\, Var(R_x)$,
where we know $Var(R_x) = Cov(R_x, R_x) = 0.16$. thus:
$Var(P) = 90{,}000 \cdot 0.16 = \boxed{14{,}400}$

b) If $(x,y,z) = (100,100,100)$, what are $E[P]$ and $Var(P)$?

i) $E[P] = E[xR_x + yR_y + zR_z] = E[100R_x + 100R_y + 100R_z]$
$= 100(E[R_x] + E[R_y] + E[R_z]) = 100(0.16 + 0.04 + 0.04)$
$= 100 \cdot 0.24 = 24$

ii) $Var(P) = Var(xR_x + yR_y + zR_z) = Var(100(R_x + R_y + R_z))$
$= 100^2\, Var(R_x + R_y + R_z)$
$= 100^2\left[\sum_{i=1}^{3} Var(R_i) + \sum_{i=1}^{3}\sum_{i \neq j} Cov(R_i, R_j)\right]$
$= 100^2\left[0.24 + 2Cov(R_x, R_y) + 2Cov(R_x, R_z) + 2Cov(R_y, R_z)\right]$
$= 10{,}000\,[0.24 + 2 \times 0.01 + 2 \times \cancel{0}^{0} + 2 \times \cancel{0}^{0}] = 10{,}000\,[0.26] = \boxed{2{,}600}$

c) Of all vectors (x,y,z) with $x \geq 0, y \geq 0, z \geq 0$ and $x+y+z=300$, which one maximizes $E[P]$?

By definition, $E[P] = E[xR_x + yR_y + zR_z]$
$$= xE[R_x] + yE[R_y] + zE[R_z]$$
$$= 0.11x + 0.10y + 0.09z.$$

So, we wish to solve the following -Linear- optimization problem:

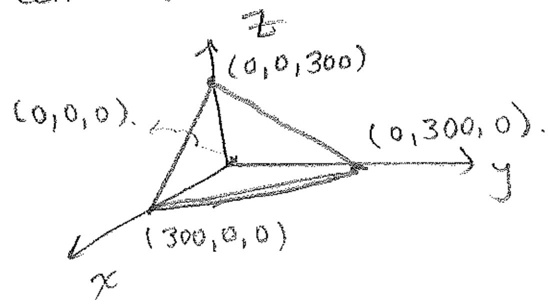
max: $0.11x + 0.10y + 0.09z$

subject to: $x \geq 0, y \geq 0, z \geq 0$

$x+y+z = 300.$

This is a Linear Programming problem. The solution is in the corners of the polyhedra formed by the constrains.

If $x=0$ then $y+z=300$
If $y=0$ then $x+z=300$
If $z=0$ then $x+y=300$



Corners	Value	
(0,0,0)	0	0
(0,0,300)	300 × 0.09	27
(0,300,0)	300 × 0.10	30
(300,0,0)	300 × 0.11	33

→ MAX value, i.e., of all vectors (x,y,z) with $x \geq 0, y \geq 0, z \geq 0$ and $x+y+z=300$, the vector $(300,0,0)$ MAXimizes $E[P]$.

d) Of all vectors (x,y,z) with $x \geq 0, y \geq 0, z \geq 0$, and $x+y+z=300$, which one maximizes $E[P] - \frac{Var(P)}{2}$? (Note, this will be the max for an exponential utility function with $b=1$).

We know that $E[P] = 0.11x + 0.10y + 0.09z$.

We need to compute $Var(P) = Var(xR_x + yR_y + zR_z)$.

$$Var(P) = Var(xR_x + yR_y + zR_z)$$

$$= Var(xR_x) + Var(yR_y) + Var(zR_z) + 2Cov(xR_x, yR_y) + 2Cov(xR_x, zR_z) + 2Cov(yR_y, zR_z)$$

$$= x^2 Var(R_x) + y^2 Var(R_y) + z^2 Var(R_z) + 2xy\,Cov(R_x, R_y) + 2xz\,\cancel{Cov(R_x, R_z)}^{\to 0} + 2yz\,\cancel{Cov(R_y, R_z)}^{\to 0}$$

$$= x^2\, 0.16 + y^2\, 0.04 + z^2\, 0.04 + 2xy\ 0.01$$

$$= 0.16x^2 + 0.04y^2 + 0.04z^2 + 0.02xy$$

Hence, $E[P] - \frac{Var(P)}{2} = 0.11x + 0.10y + 0.09z - 0.08x^2 - 0.02y^2 - 0.02z^2 - 0.01xy$

So, we want to solve the following non-linear, constrained optimization:

$$\max: \quad 0.11x + 0.10y + 0.09z - 0.08x^2 - 0.02y^2 - 0.02z^2 - 0.01xy$$

s.t. $x \geq 0,\ y \geq 0,\ z \geq 0$

$x + y + z = 300. \Rightarrow z = 300 - x - y.$

Replace the constrain $x+y+z=300$ in the function to be optimazed:

$$0.11x + 0.10y + 0.09(300-x-y) - 0.08x^2 - 0.02y^2 - 0.02(300-x-y)^2 - 0.01xy = f(x,y)$$

SET GRADIENT EQUAL to ZERO:

$$\frac{\partial f}{\partial x} = 0.11 - 0.09 - 0.16x + 0.04(300-x-y) - 0.01y$$

$$\frac{\partial f}{\partial x} = 12.02 - 0.20x - 0.05y = 0 \Rightarrow 0.20x = 12.02 - 0.05y \Rightarrow x = 60.1 - 0.25y \quad ①$$

$$\frac{\partial f}{\partial y} = 0.10 - 0.09 - 0.04y + 0.04(300-x-y) - 0.01x$$

$$\frac{\partial f}{\partial y} = 12.01 - 0.08y - 0.05x = 0 \Rightarrow 0.05x = 12.01 - 0.08y \Rightarrow x = 240.2 - 1.6y \quad ②$$

$① = ② \Rightarrow 60.1 - 0.25y = 240.2 - 1.6y \Rightarrow 180.1 = 1.35y \Rightarrow \boxed{y = 133.407}$

$x = 60.1 - 0.25(133.407) \Rightarrow \boxed{x = 26.7481}$ $\quad z = 300 - x - y \Rightarrow \boxed{z = 139.844852}$