M451 - Enrique Areyan - Spring 2015 - HW 9

Chapter 9:

(9.14) If the beta of a stock is .80, what is the expected rate of return of that stock if the expected value of the market's rate of return is .07 and the risk-free interest rate is 5%? What if the risk-free interest rate is 10%? Assume the CAPM.

Sol: According to CAPM: $r_i = r_f + \beta_i (r_m - r_f)$. Hence,

(a) $\beta = .8$, $r_i = ?$, $r_m = .07$, $r_f = .05$,

$r_i = .05 + .8(.07 - .05) = .05 + .8(.02) = .05 + .016 = \boxed{0.066}$

(b) $\beta = .8$, $r_i = ?$, $r_m = .07$, $r_f = .1$,

$r_i = .1 + .8(.07 - .1) = .1 + .8(-.03) = .1 - .024 = \boxed{0.076}$

(9.15) If β_i is the beta of stock i for $i = 1, \ldots, k$, what would be the beta of a portfolio in which α_i is the fraction of one's capital that is used to purchase stock i ($i = 1, \ldots, k$)?

Sol: We know that; for a given security i:

$$\beta_i = \frac{Cov(R_i, R_m)}{Var(R_m)} \Rightarrow Cov(R_i, R_m) = \beta_i Var(R_m)$$

Let β_i be the beta of stock i and β_p be the beta of the portfolio. The rate of return of the portfolio is $R_p = \sum_i \alpha_i R_i$. Thus,

$$\beta_p = \frac{Cov(R_p, R_m)}{Var(R_m)} = \frac{Cov(\sum_i \alpha_i R_i, R_m)}{Var(R_m)} = \frac{\sum_i \alpha_i Cov(R_i, R_m)}{Var(R_m)}$$

$$= \frac{\sum_i \alpha_i \beta_i Var(R_m)}{Var(R_m)} = \frac{\cancel{Var(R_m)} \sum_i \alpha_i \beta_i}{\cancel{Var(R_m)}} = \boxed{\sum_i \alpha_i \beta_i}$$

The beta of the portfolio is the weighted sum of individual betas.

(9.16). A single-factor model supposes that R_i, the one-period rate of return of a specified security, can be expressed as

$$R_i = a_i + b_i F + e_i,$$

where F is a random variable (called the "factor"), e_i is a normal random variable with mean 0 that is independent of F, and a_i and b_i are constants that depend on the security. Show that CAPM is a single-factor model, and identify a_i, b_i and F.

<u>Sol</u>: Let R_i be the one-period rate of return of a specified security i
R_m be the one-period rate of return of the entire market
r_f the risk-free interest rate

The CAPM states:

$$R_i = r_f + \beta_i(R_m - r_f) + e_i, \text{ where } e_i \sim \text{Normal}(0, \sigma_i^2).$$

And e_i is independent of R_m.

If we rewrite CAPM like:

$$R_i = r_f + \beta_i R_m - \beta_i r_f + e_i = (1-\beta_i) r_f + \beta_i R_m + e_i, \text{ then}$$

we can see that CAPM is a single-factor model with

$a_i = (1-\beta_i) r_f$, $b_i = \beta_i$, and $F = R_m$, where e_i is independent of F.

<u>Chapter 10</u>:

(10.3) Let $X(n,p)$ denote a binomial random variable with parameters n and p. If $p_1 \geq p_2$, show that $X(n,p_1) \geq_{s.t} X(n,p_2)$.

<u>Pf</u>: We will show this result by coupling, i.e., find random variables X' and Y' s.t. X' has the same distribution as $X(n,p_1)$ and Y' has the same distribution as $X(n,p_2)$ s.t. $X' > Y'$ always.

Consider:

$$Y_{1i} = \begin{cases} 0 & \text{with prob. } p_1 \\ 1 & \text{with prob. } 1-p_1 \end{cases}$$

$$Y_{2i} = \begin{cases} 0 & \text{with prob. } p_2/p_1 < 1 \text{ since } p_1 > p_2. \\ 1 & \text{with prob. } 1-\frac{p_2}{p_1}. \end{cases}$$

where Y_{1i}'s are indep. Y_{2i}'s are indep. Y_{1i}'s are inde. of Y_{2i}'s.

then, $X(n,p_1) = \sum_{i=1}^{n} Y_{1i}$ and $X(n,p_2) = \sum_{i=1}^{n} Y_{1i} \cdot Y_{2i}$

the second statement is true because

$$Y_{1i} \cdot Y_{2i} = \begin{cases} 1 & \text{with prob. } p_1\left(\frac{p_2}{p_1}\right) = p_2 \text{ (in case both } Y_{1i} \text{ \& } Y_{2i} \text{ are 1)} \\ 0 & \text{with prob } 1-\left(p_1\left(\frac{p_2}{p_1}\right)\right) = 1-p_2. \end{cases}$$

But then, $X(n,p_1)$ has the same distribution as $\sum_{i=1}^{n} Y_{1i}$ and
$X(n,p_2)$ " " " " as $\sum_{i=1}^{n} Y_{1i} \cdot Y_{2i}$.

Clearly, $\forall i: Y_{1i} \cdot Y_{2i} \leq Y_{1i}$, so that $\sum_{i=1}^{n} Y_{1i} \cdot Y_{2i} \leq \sum_{i=1}^{n} Y_{1i}$.

this shows that $X(n,p_1) \geq_{s.t} X(n,p_2)$.

(10.4) If $X_i \sim Normal(\mu_i, \sigma^2)$, for $i=1,2$.
Show that $X_1 \geq_{lr} X_2$ when $\mu_1 \geq \mu_2$.

Pf: $X \geq_{lr} Y$ if $\frac{f(x)}{g(x)}$ is increasing in x over the region where either $f(x)$ or $g(x)$ is greater than 0, where $f(x)$ is the density of X and $g(x)$ is the density of Y.

Let $X_1 \sim Normal(\mu_1, \sigma^2)$ AND $X_2 \sim Normal(\mu_2, \sigma^2)$ AND $\mu_1 \geq \mu_2$. then
Let f_1 be the density of X_1 and f_2 the density of X_2

$$\frac{f_1(x)}{f_2(x)} = \frac{\cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma^2}\right)}{\cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} \exp\left(-\frac{(x-\mu_2)^2}{2\sigma^2}\right)} = \exp\left(\frac{-(x-\mu_1)^2}{2\sigma^2} + \frac{(x-\mu_2)^2}{2\sigma^2}\right)$$

$$= \exp\left(\frac{\cancel{-x^2} + 2\mu_1 x - \mu_1^2 + \cancel{x^2} - 2x\mu_2 + \mu_2^2}{2\sigma^2}\right) = \exp\left(\frac{2x(\mu_1-\mu_2) + \mu_2^2 - \mu_1^2}{2\sigma^2}\right)$$

Hence, $\frac{f_1(x)}{f_2(x)} = \exp\left(\frac{2(\mu_1-\mu_2)x + \mu_2^2 - \mu_1^2}{2\sigma^2}\right)$, so the ratio is of

the form $\exp(ax+b)$, where $a \geq 0$ only because $\mu_1 \geq \mu_2$.

this function is increasing for $x \in \mathbb{R}$, showing that $X_1 \geq_{lr} X_2$

(10.5) Let X_i be an exponential random variable with density function $f_i(x) = \lambda_i e^{-\lambda_i x}$, $i=1,2$. If $\lambda_1 \leq \lambda_2$, show that $X_1 \geq_{lr} X_2$.

Pf: Compute the likelihood ratio:

$$\frac{f_1(x)}{f_2(x)} = \frac{\lambda_1 e^{-\lambda_1 x}}{\lambda_2 e^{-\lambda_2 x}} = \frac{\lambda_1}{\lambda_2} e^{-\lambda_1 x + \lambda_2 x} = \frac{\lambda_1}{\lambda_2} e^{x(\lambda_2 - \lambda_1)} \longrightarrow \boxed{a e^{bx},\ a,b > 0}$$

increasing function.

this is an increasing function provided $\lambda_1 \leq \lambda_2 \Rightarrow \lambda_2 - \lambda_1 \geq 0$.

Note that $\lambda_1, \lambda_2 > 0$.

(10.6) Let X_i be a Poisson random variable with mean λ_i. If $\lambda_1 \geq \lambda_2$, show that $X_1 \geq_{lr} X_2$.

Pf: Poisson is a discrete random variable, so to test the lr, we want to show that $\frac{P(X_1=n)}{P(X_2=n)}$ is increasing in n, for $n \in \{0,1,2,\dots\}$.

$$\frac{P(X_1=n)}{P(X_2=n)} = \frac{\frac{\lambda_1^n}{n!} e^{-\lambda_1}}{\frac{\lambda_2^n}{n!} e^{-\lambda_2}} = \frac{\lambda_1^n \cancel{n!} e^{\lambda_2-\lambda_1}}{\lambda_2^n \cancel{n!}} = \left(\frac{\lambda_1}{\lambda_2}\right)^n e^{\lambda_2-\lambda_1}.$$

this function is of the form $b \cdot a^n$, where $b \geq 0$ indep. of λ_1, λ_2, but $a > 0$ only because $\lambda_1 \geq \lambda_2$ so $\frac{\lambda_1}{\lambda_2} \geq 1$.

8. Find an example showing that $X \geq_{st} Y$ does not always imply $X \geq_{lr} Y$. Can you find an example of two random variables X, Y such that $X \geq_{st} Y$ but $Y \geq_{lr} X$?

<u>Sol</u>: Consider the following two random variables: (discrete):

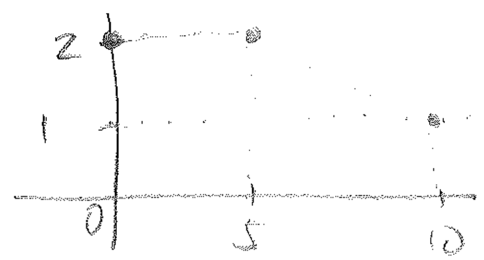
X with pmf: $f(x) = \begin{cases} 0.1 & x=0 \\ 0.1 & x=5 \\ 0.9 & x=10 \end{cases}$, 0 if $x \notin \{0,5,10\}$

Y with pmf: $g(x) = \begin{cases} 0.05 & x=0 \\ 0.05 & x=5 \\ 0.9 & x=10 \end{cases}$, 0 if $x \notin \{0,5,10\}$.

Clearly $X \geq_{st} Y$. $(P(X>t) \geq P(Y>t))$.

But it is <u>not</u> the case that $X \geq_{lr} Y$, because.

$$\frac{P(X=k)}{P(Y=k)} = \begin{cases} 0.1/0.05 = 2 & \text{if } k=0 \text{ or } k=5 \\ 0.9/0.9 = 1 & \text{if } k=10, \end{cases}$$



So the ratio is decreasing.

Moreover, this is an example of a pair of R.V s.t $X \geq_{st} Y$ but $Y \geq_{lr} X$. That $X \geq_{st} Y$ is clear. Now,

$$\frac{P(Y=k)}{P(X=k)} = \begin{cases} \frac{0.05}{0.1} = 1/2 & \text{if } k=0 \text{ or } k=5 \\ 0.9/0.9 = 1 & \text{if } k \notin \{0,5,10\} \end{cases}$$ (increasing on k) (for $k=0, 5$ or 10)

which shows $Y \geq_{lr} X$.