M451/551 Quiz 2

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1. Let $\{X(t), t \geq 0\}$ be a Brownian motion process with drift parameter μ and variance parameter σ^2. Assume that $X(0) = 0$, and let T_y be the first time that the process is equal to y. For $y > 0$, show that

$$P(T_y < \infty) = \begin{cases} 1 & \mu \geq 0 \\ e^{2y\mu/\sigma^2} & \mu \not> 0. \quad \mu < 0 \end{cases}$$

Let $M = \max_{0<t<\infty} X(t)$ be the maximal value ever attained by the process, and conclude from the preceding that, when $\mu < 0$, M has an exponential distribution with rate $-2\mu/\sigma^2$.

You may use the following formula:

$$P(T_y < t) = e^{2y\mu/\sigma^2}\bar{\Phi}\left(\frac{y+\mu t}{\sigma\sqrt{t}}\right) + \bar{\Phi}\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right)$$

$$P(T_y < \infty) = \lim_{t\to\infty} P(T_y < t) = \lim_{t\to\infty} e^{2y\mu/\sigma^2}\bar{\Phi}\left(\frac{y+\mu t}{\sigma\sqrt{t}}\right) + \bar{\Phi}\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) \quad \leftarrow \text{By def. of } P(T_y<t)$$

$$= e^{2y\mu/\sigma^2}\lim_{t\to\infty}\bar{\Phi}\left(\frac{y+\mu t}{\sigma\sqrt{t}}\right) + \lim_{t\to\infty}\bar{\Phi}\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) \quad \ldots \text{By properties of limit.}$$

Now, the quantity $\frac{y+\mu t}{\sigma\sqrt{t}} = \frac{y}{\sigma\sqrt{t}} + \frac{\mu t}{\sigma\sqrt{t}} = \frac{y}{\sigma\sqrt{t}} + \frac{\mu}{\sigma}\sqrt{t} \longrightarrow 0 + \frac{\mu}{\sigma}\infty$ as $t\to\infty$.

So the sign $(\pm\infty)$, depends on the sign of $\frac{\mu}{\sigma}$. Since we assume $\sigma > 0$, the sign will depend only on μ. Hence, if $\mu > 0 \Rightarrow \frac{y+\mu t}{\sigma\sqrt{t}} \to -\infty$ and $\frac{y-\mu t}{\sigma\sqrt{t}} \to +\infty$. Since $\bar{\Phi}$ is continuous, we have:

$\mu = 0$ ...

If $\mu > 0$: $P(T_y < \infty) = e^{2y\mu/\sigma^2}\bar{\Phi}(-\infty) + \bar{\Phi}(\infty) = 1$

If $\mu < 0$: $P(T_y < \infty) = e^{2y\mu/\sigma^2}\bar{\Phi}(+\infty) + \bar{\Phi}(-\infty) = e^{2y\mu/\sigma^2}$

} this shows the first result.

Now, Let $M = \max_{0<t<\infty} X(t)$. We know that the event that $M > t$ is equivalent to $T_y < \infty$ Hence, by preceding result:

$P(M > t) = P(T_y < \infty) = e^{2y\mu/\sigma^2}$, provided $\mu < 0$. But then,

$P(M \leq t) = 1 - P(M > t) = 1 - e^{2y\mu/\sigma^2} = 1 - e^{\frac{2\mu y}{\sigma^2}}$, which shows that M has an exponential distribution, since its cdf is given as above

↑ this is the cdf

(Problem #2 is on the other side.)

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2. Assuming that

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}x^2+ax+b}dx = e^{\frac{1}{2}a^2+b},$$

show that if $S(t) = S(0)e^{X(t)}$ for a Brownian motion with $X(0) = 0$ and drift μ and variance σ^2, then

$$\mathbb{E}[S(t)] = S(0)e^{\mu t + t\sigma^2/2}.$$

You may use that the PDF for a normal random variable with mean μ and variance σ^2 is:

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$$

$$E[S(t)] = E[S(0)e^{X(t)}] = S(0)\,E[e^{X(t)}]$$

To compute the expection we use the definition

$$E[e^{X(t)}] = \int_{\mathbb{R}} e^x \,\mathrm{pdf}_x\, dx = \int_{\mathbb{R}} e^x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}} e^{x - \frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}} e^{x - \frac{(x^2-2\mu x+\mu^2)}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}} e^{\frac{2\sigma^2 x - x^2 + 2\mu x - \mu^2}{2\sigma^2}} dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}} e^{\frac{-x^2+x(2\mu+2\sigma^2)-\mu^2}{2\sigma^2}} dx$$

Using the given fact, this integral becomes

$$= \frac{1}{\sigma\sqrt{2\pi}}\cdot\sqrt{2\pi}\cdot\sigma\; e^{\frac{1}{2}(\mu+\sigma^2)^2-\mu^2}$$

$$= e^{\mu t + t\sigma^2/2},$$ And finally,

$$E[S(t)] = S_0\,E[e^{X(t)}] = \boxed{S_0\, e^{\mu t + \frac{t\sigma^2}{2}}}$$

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