M451/551 Quiz 7

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Name: Enrique Areyan.

You do not need to simplify numerical expressions.

1. The current price of a security is S_0, and assume S_t follows a G.B.M. Consider an investment whose cost is S_0 and whose payoff at time 1 is, for a specified choice of β satisfying $0 < \beta < e^r - 1$, given by

$$\text{return} = \begin{cases} (1+\beta)S_0 & \text{if } S_1 \leq (1+\beta)S_0, \\ (1+\beta)S_0 + \alpha(S_1 - (1+\beta)S_0) & \text{if } S_1 \geq (1+\beta)S_0. \end{cases}$$

Determine the value of α if this investment (whose payoff is both uncapped and always greater than the initial cost of the investment) is not to give rise to an arbitrage.

The risk-neutral G.B.M. value of the investment is given by:

$$E[(1+\beta)S_0 + \alpha(S_1 - (1+\beta)S_0)^+] = E[(1+\beta)S_0] + \alpha E[(S_1 - (1+\beta)S_0)^+]$$

(by linearity of expectation)

$$= (1+\beta)S_0 + \alpha e^r C(S_0, 1, (1+\beta)S_0, \sigma, r)$$

(since $(1+\beta)S_0$ is a constant AND by definition of B-S).

Note that this investment costs S_0, same as the current price of the security. Therefore, in order to not give rise to arbitrage they should yield the same payoff.

The payoff from security @ time 1 is $S_0 e^r$ (by selling it at time 0 and investing S_0). Therefore, the value of the investment is equal to $S_0 e^r$:

$$S_0 e^r = (1+\beta)S_0 + \alpha e^r C(S_0, 1, (1+\beta)S_0, \sigma, r)$$

Now solve for α:

$$\boxed{\alpha = \frac{S_0(e^r - (1+\beta))}{e^r C(S_0, 1, (1+\beta)S_0, \sigma, r)}}$$

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(Problem #2 is on the other side.)

2. An option on an option, sometimes called a compound option, is specified by the parameter pairs (K_1, T_1) and (K, T), where $T_1 < T$. The holder of such a compound option has the right to purchase, for the amount K_1, a (K, T) call option on a specified security. This option to purchase the (K, T) call option can be exercised any time up to time T_1.

   (a) Argue that the option to purchase the (K, T) call option would never be exercised before its expiration time T_1. (You are not required to prove an arbitrage portfolio.)

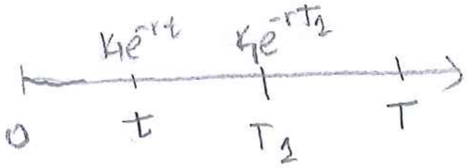
   (b) Argue that the option to purchase the (K, T) call option should be exercised if and only if $S_{T_1} \geq x$, where x is the solution of

   $$K_1 = C(x, T - T_1, K, , r),$$

   $C(S_0, T, K, , r)$ is the BlackScholes formula, and S_{T_1} is the price of the security at time T_1.

   (c) Argue that there is a unique value x that satisfies the preceding identity.

(a)



If you exercise the option at time t, s.t. $t < T_1$, then you pay $K_1 e^{-rt}$. However, if you wait until T_1, then you pay $K_1 e^{-rT_1}$. Since $t < T_1$, it follows $K_1 e^{-rt} > K_1 e^{-rT_1}$, So, a dominating strategy is to wait until T_1 and pay less.

(b) ($\Leftarrow$) Suppose $S_{T_1} \geq x$, where x is the sol. of $K_1 = C(x, T-T_1, K, \sigma, r)$ then, you should exercise the (K,T) call option because doing so will provide a positive balance. this follows b/c $C(x, T-T_1, K, r)$ is the value of a call option with initial price x and expiration $T-T_1$, so the value S_{T_1} is at least the value of said call option.

($\Rightarrow$) Suppose the call option should be exercised, then clearly the value of the stock @ time T_1 should be at least as big as the initial price of the call option.

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(c) that there is a unique value follows directly from previously proven fact that B-S is an (strictly) increasing function of the initial price, in this case $C(x, T-T_1, K, \sigma)$ is an increasing function of x, so only one value will satisfy $K_1 = C(x, T-T_1, K, \sigma, r)$.