M451/551 Quiz 8

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Name: Enrique Areyan

You do not need to simplify numerical expressions.

1. The utility function of an investor is $u(x) = 1 - e^{-x}$. The investor must choose one of two investments. If his fortune after investment 1 is a random variable with density function $f_1(x) = e^{-x}$, $x > 0$, and his fortune after investment 2 is a random variable with density function $f_2(x) = 1/2$, $0 < x < 2$, which investment should he choose?

Let X = fortune after investment 1

Y = fortune after investment 2.

He should choose the investment $\max\{E[u(X)], E[u(Y)]\}$.

$$E[u(X)] = \int_0^\infty (1-e^{-x})e^{-x}\,dx = \int_0^\infty e^{-x}\,dx - \int_0^\infty e^{-2x}\,dx$$

$$= \left[-e^{-x}\right]_0^\infty - \left[-\frac{e^{-2x}}{2}\right]_0^\infty$$

$$= \left\{\lim_{t\to\infty}\left[-e^{-t}\right] - \left[-e^{0}\right]\right\} - \left\{\lim_{t\to\infty}\left[-\frac{e^{-2t}}{2}\right] - \left[-\frac{e^{0}}{2}\right]\right\}$$

$$= 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

$$E[u(Y)] = \int_0^2 (1-e^{-y})\frac{1}{2}\,dy = \frac{1}{2}\left[\int_0^2 1\,dy - \int_0^2 e^{-y}\,dy\right]$$

$$= \frac{1}{2}\left[2 - \left[-e^{-y}\right]_0^2\right]$$

$$= \frac{1}{2}\left[2 + \left[e^{-2} - e^{0}\right]\right]$$

$$= \frac{1}{2}\left[1 + e^{-2}\right]$$

Since $e^{-2} > 0 \Rightarrow 1 + e^{-2} > 1$, so $\underbrace{\frac{1}{2}(1+e^{-2})}_{E[u(Y)]} > \underbrace{\frac{1}{2}}_{E[u(X)]}$

He should choose investment 2.

(Problem #2 is on the other side.)

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2. In the portfolio selection problem, show that the percentage of ones wealth that should be invested in each security when attempting to maximize $E[\log(W_1)]$ does not depend on the amount of initial wealth W_0.

We know that:

$$W_1 = \alpha_n X_n \alpha_{n-1} X_{n-1} \cdots \alpha_1 X_1 \alpha_0 X_0 W_0,$$

where W_0 is the amount of initial wealth.

α_i is the % to be invested in investment i

X_i is investment i.

thus,

$$\log(W_1) = \log(\alpha_n X_n \alpha_{n-1} X_{n-1} \cdots \alpha_1 X_1 \alpha_0 X_0 W_0)$$

$$= \log(W_0) + \sum_{i=0}^{n} \log(\alpha_i X_i)$$

Taking expectation

$$E[\log(W_1)] = E\left[\log(W_0) + \sum_{i=0}^{n} \log(\alpha_i X_i)\right] \ldots \text{ by def. of } \log(W_1)$$

$$= E[\log(W_0)] + E\left[\sum_{i=0}^{n} \log(\alpha_i X_i)\right] \ldots \text{ by linearity of expectation}$$

$$= \log(W_0) + E\left[\sum_{i=1}^{n} \log(\alpha_i X_i)\right] \ldots \text{ since } \log(W_0) \text{ is a constant.}$$

So, we want to solve the optimization

$$\underset{\alpha_1, \ldots, \alpha_n}{\text{maximize}} \quad \log(W_0) + E\left[\sum_{i=1}^{n} \log(\alpha_i X_i)\right]$$

As soon as we take derivatives $\log(W_0)$ will vanish.

Hence, the maximum does not depend on initial wealth W_0

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