M451 - Enrique Areyan - Spring 2015 - Study Guide

Let X be a random variable. Then,
S_X = set of all possible values of X. (Usually $S_X \subseteq \mathbb{R}$).

For independent R.V. $E[X \cdot Y] = E[X] \cdot E[Y]$

Def: $Var(X) = E[(X-E[X])^2] = E[X^2] - E[X]^2$

If X, Y are independent, then $Var(X+Y) = Var(X) + Var(Y)$

$Var(aX) = a^2 Var(X)$. constant come out squared

Def: $Cov(X,Y) = E[(X-E[X])(Y-E[Y])]$,

If X, Y are independent, then $Cov(X,Y) = 0$

Covariance is bilinear and $Cov(c, Y) = 0$, for c a constant

Def: Correlation of variables $\rho(X,Y) = Cov(X,Y) / \sqrt{Var(X)}\sqrt{Var(Y)}$.

Also, $-1 \leq \rho(X,Y) \leq 1$.

Def: Conditional Expectation: $E[X \mid Y=y] = \sum_{x \in S_X} x \cdot P(X=x \mid Y=y)$.

claim: $E[E[X \mid Y]] = \sum E[X \mid Y=y] P(Y=y) = E[X]$.

Normal Random Variables: $X_{\mu,\sigma}$ is a N.R.V. if f_X (pdf) is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \text{for } x \in \mathbb{R}.$$

μ = mean of X
$\sigma = \sqrt{Var(X)}$ = stdev., $\sigma > 0$, $\mu \in \mathbb{R}$.

Define: For $X_{0,1}$, the standard N.R.V. define

$$\Phi(x) := P(X_{0,1} \leq x) = \int_{-\infty}^{x} f_{X_{0,1}}(t)\,dt \text{ ; the c.d.f.}$$

From this it follows: $\Phi(-x) = 1 - \Phi(x)$ [think of $1 = \Phi(\infty)$]. Also

$$P(a \leq X_{0,1} \leq b) = \Phi(b) - \Phi(a).$$

FACT: If X is a N.R.V., then so is $aX + b$, where b is viewed as $X_{b,0}$

To normalize a N.R.V X, do: $X_{0,1} = \frac{X_{\mu,\sigma} - \mu}{\sigma}$

FACT: If X_1, X_2 are 2 independent N.R.V., then $X_1 + X_2 \sim Normal(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

CENTRAL LIMIT THEOREM: Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables. Let $S_n = \sum_{i=1}^{n} X_i$. Then,

$$\lim_{n \to \infty} \frac{S_n - n\mu}{\sigma\sqrt{n}} = X_{0,1}$$, where the convergence speed depends on the pdf.

Def: A r.v. Y is log-normal if it has the form $Y = e^{X_{\mu,\sigma}}$, where $X_{\mu,\sigma}$ is normal. Then, $E[Y] = e^{\mu + \frac{\sigma^2}{2}}$

$$Var[e^{X_{\mu,\sigma}}] = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}.$$

<u>Brownian Motion</u>: $X(t)$ is a B.M. with drift μ and variance 6^2 if:
1) $X(0)$ is a fixed constant; 2) $\forall s,t: X(s+t)-X(t) \sim Normal(\mu s, 6^2 s)$.

<u>THEOREM</u>: with probability 1, $X(t)$ is continuous.

<u>Def</u>: A <u>discrete model</u> for B.M $X_\Delta(t)$, for $\Delta > 0$, satisfies:

1) $X_\Delta(0) = $ constant; 2) $X_\Delta(t+\Delta) = \begin{cases} X_\Delta(t) + 6\sqrt{\Delta} & \text{with probability } p \\ X_\Delta(t) - 6\sqrt{\Delta} & \text{with probability } 1-p \end{cases}$

AND $X_\Delta(t+s) = X(t)$ if $s < \Delta$ (discrete jumps).

where $p = \frac{1}{2}\left(1 + \frac{\mu}{6}\sqrt{\Delta}\right)$

<u>Def</u>: $X_i = \begin{cases} +1 & \text{if } X_\Delta(i\Delta) > X_\Delta((i-1)\Delta) \\ -1 & \text{if } X_\Delta(i\Delta) < X_\Delta((i-1)\Delta) \end{cases} \Rightarrow X_\Delta(t+n\Delta) = X(t) + 6\sqrt{\Delta}\sum_{i=1}^{n} X_i$

the X_i are independent by Markov Property.

$X(t+s)$ has $\lfloor \frac{s}{\Delta} \rfloor$ changes since time t. $\Rightarrow X_\Delta(t+s) - X_\Delta(t) = 6\sqrt{\Delta}\sum_{i=1}^{\lfloor s/\Delta \rfloor} X_i$

Then, $E[X_i] = 1 \cdot p + (-1)(1-p) = 2p-1 = \frac{\mu}{6}\sqrt{\Delta}$ $\quad$ as $\Delta \to 0$

$Var[X_i] = E[X_i^2]^{1} - E[X_i]^2 = 1 - \frac{\mu^2}{6^2}\Delta$

So, $E[X_\Delta(t+s) - X_\Delta(t)] = E\left[6\sqrt{\Delta}\sum_{i=1}^{\lfloor s/\Delta \rfloor} X_i\right] = 6\sqrt{\Delta}\sum_{i=1}^{\lfloor s/\Delta \rfloor} E[X_i] = 6\sqrt{\Delta}\left\lfloor \frac{s}{\Delta}\right\rfloor \frac{\mu}{6}\sqrt{\Delta} = s \cdot \mu + O(\Delta)$

$Var[X_\Delta(t+s) - X_\Delta(t)] = Var\left[6\sqrt{\Delta}\sum_{i=1}^{\lfloor s/\Delta \rfloor} X_i\right] = 6^2\Delta \sum_{i=1}^{\lfloor s/\Delta \rfloor} Var[X_i] = 6^2\Delta\left\lfloor\frac{s}{\Delta}\right\rfloor\left(1 - \frac{\mu^2}{6^2}\Delta\right)$ $\quad 0$ as $\Delta \to 0$

$= 6^2 s\left(1 - \frac{\mu^2}{6^2}\Delta\right) + O(\Delta)$ $\quad$ as $\Delta \to 0$

So, $X_\Delta(s+t) - X_\Delta(t) \to$ random variable with mean $\mu \cdot s$ and variance $6^2 \cdot s$.

Finally, By CLT: $\left(X_\Delta(s+t) - X_\Delta(t) - \mu \cdot s\right) / \sqrt{\lfloor \frac{s}{\Delta} \rfloor} \cdot 6 \cdot \sqrt{s} \xrightarrow{CLT} X_{0,1}$

So $X_\Delta(s+t) - X_\Delta(t) \xrightarrow{\Delta \to 0} X_{\mu s, 6^2 s}$ AND $X_\Delta(t) \xrightarrow{\Delta \to 0} X(t)$ continuous B.M.

<u>THEOREM</u>: Given that $X(t) = x_t \in \mathbb{R}$, the conditional prob. of $X(s)$, $0 \le s \le t$ is the same for all values of μ (depends on 6)

the B.M has 2 flaws when dealing with stocks:
(1) $P_r\{X(t) < 0\} > 0$ [but stocks are always $\geq \$0$] AND (2) the prob. that a \$1 stock loses \$1 seems less than the prob. that \$1000 $\to$ \$999

<u>Geometric Brownian Motion</u>: A stochastic process $S(t)$ with drift μ and variance 6^2 if $\log(S(t))$ is a B.M. with those parameters.

So, $S(t) = e^{X(t)} \Rightarrow \log\left[\frac{S(t+s)}{S(t)}\right] = \log(S(t+s)) - \log(S(t)) \sim Normal(\mu s, 6^2 s)$

Note $S(t) \geq 0$ & the ratios of price differences follow a normal dist. (rates change gain/loss %)

6 is called <u>volatility</u>. We normalize $S(t)$ to be $S(t) = S_0 e^{X(t)}$, where $S(0) = S_0$

So $X(0) = 0$ [B.M. starts at 0].

$E[e^X] = e^{E[X] + Var(X)/2}$ $\qquad E[S(t)] = S_0 e^{\mu t + t 6^2/2}$

If $X \sim Normal$

Discrete Approximation of Geometric B.M

Break into Δ-time increments as we did for X_Δ.

$S_\Delta(t+\Delta) = S_\Delta(t)\, e^{X_\Delta(t+\Delta) - X(t)}$ Since $X_\Delta(t+\Delta) - X(t) = \pm 6\sqrt{\Delta}$, it follows:

$$\frac{S_\Delta(t+\Delta)}{S_\Delta(t)} = \begin{cases} e^{6\sqrt{\Delta}} & \text{prob } p \\ e^{-6\sqrt{\Delta}} & \text{prob } 1-p \end{cases}$$ as $\Delta \to 0$, $X_\Delta(t) \to$ B.M. $S_\Delta(t) \to$ G.B.M.

The Maximum Variable: Given a B.M. $X(t)$ with drift μ and var 6^2,

Define: the max value of B.M. $M(t) = \max_{0 \le s \le t} X(s)$.

THM: the conditional distribution of $M(t)$ given $x(t) = x_t$ is,

for $y > x_t$: $P(M(t) \ge y \mid x(t) = x_t) = e^{-2y(y - x_t)/t6^2}$

Corollary: For $y > 0$, $P(M(t) > y) = e^{2y\mu 6^2}\left(1 - \Phi\left(\frac{\mu t + y}{6\sqrt{t}}\right)\right) + \left(1 - \Phi\left(\frac{y - \mu t}{6\sqrt{t}}\right)\right)$

Interest Rates: You borrow P dollars at a (simple) compounding interest rate $r > 0$ per time T. This means that at time T you owe

$$P + rP = P(1+r) \text{ dollars}$$

If we consider time t, we will owe approx. $P(1+r)^{t/T}$

If $T < 1$ year, the effective 1 yr. interest rate is $r_{eff} = (1+r)^{\frac{1}{T}} - 1$

Because: $P(1+r)^{1/T} = P(1 + r_{eff}) \Rightarrow r_{eff} = (1+r)^{\frac{1}{T}} - 1$

EX: Nominal yearly interest rate r_0 is compounded 6 times, then $r_{eff} = (1 + \frac{r_0}{6})^6 - 1$. If we compound n times $r_{eff} = (1 + \frac{r_0}{n})^n - 1$

taking $n \to \infty$, $r_{eff} = e^{r_0} - 1$ [there is a limit to how much compounding works]

Present Value Analysis. Let $\mathbf{a} = (a_1, a_2, \dots, a_k)$ be an income stream. Note that payment a_i is currently worth $a_i(1+r)^{-i}$. Hence

$$PV(\mathbf{a}) = a_1(1+r)^{-1} + a_2(1+r)^{-2} + \dots + a_k(1+r)^{-k} = \sum_{i=1}^{k} a_i(1+r)^{-i}$$

Note that to keep the present value constant if early payments are reduced, then later payments must be increased disproportionately to account for the exponentially smaller contribution of later payments

EX: $\mathbf{a} = (P, P, P, \dots) \Rightarrow PV(\mathbf{a}) = \sum_{i=0}^{\infty} P(1+r)^{-i} = P\sum_{i=0}^{\infty}(1+r)^{-i} = P\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i$

$= P\left[\frac{1}{1 - \frac{1}{1+r}}\right] = P\left[\frac{1+r}{r}\right]$ [Note as $r \to \infty$, $PV(\mathbf{a}) \to P$, interest rate so high future payments worth nothing]

Recall $1+x+x^2+\cdots+x^n = \dfrac{1-x^{n+1}}{1-x}$ [To verify, multiply through by $1-x$)

<u>CASH Flows</u>: Given $\mathbf{b} = b_1, b_2, \ldots, b_n$ & $\mathbf{c} = c_1, c_2, \ldots, c_n$,
when is $PV(\mathbf{b}) \geq PV(\mathbf{c})$, for any $r > 0$?

(*) If $b_i \geq c_i \ \forall i = 1, \ldots, n \implies b_i \alpha^i \geq c_i \alpha^i$. Define $B_i = \sum_{j=1}^{i} b_j$.
Then, $B_i \geq C_i$ for $i = 1, \ldots, n$ and Then $PV(\mathbf{b}) \geq PV(\mathbf{c})$.

- But we can do better -

(**) <u>Proposition</u>: If $B_n \geq C_n$ and $\sum_{i=1}^{k} B_i \geq \sum_{i=1}^{k} C_i \quad \forall k = 1, \ldots, n$.
then $PV(\mathbf{b}) \geq PV(\mathbf{c})$

<u>RATE of return</u>: <u>Def</u>: R.oR. of an investment A that returns B after 1 period of time is the interest rate that would make $PV(B) = A$

$\frac{b}{1+r} = a \implies r = \frac{b-a}{a} = \frac{b}{a} - 1$

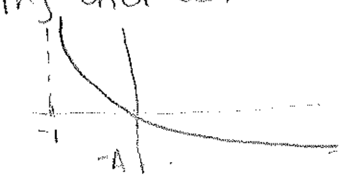
If we have initial investment of A and retur stream $\mathbf{B} = b_1, b_2, \ldots, b_n$,
s.t. $PV(\mathbf{B}) = A$.
Then the R.oR. is the interest rate r s.t. $\sum_{i=1}^{n} B_i (1+r)^{-i}$, Then the R.oR.

As a function or r: $P(r) = -A + \sum_{i=1}^{n} B_i (1+r)^{-i}$

r^* is the value s.t. $P(r^*) = 0$., $r > -1$.

Note that this is in general not well defined. However, if $A > 0$ and all $B_i > 0$ then $P(r)$ is monotonically decreasing and continuous since $\frac{1}{1+r}$ is. Also

as $r \to -1^+$; $P(r) \to \infty$
as $r \to \infty$; $P(r) \to -A$



So, there is such r^*. This only works for $A > 0$ and $b_i \geq 0$, where one $b_j > 0$.

Note: if $r > r^*$, then $P(r) < 0$ (neg. return)
if $r < r^*$, then $P(r) > 0$ (pos. return)

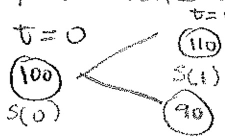
<u>Continously Varying Interest Rates</u>

$r(s)$ = interest rate at time s called <u>spot</u> or instantaneous interest rate.
$D(t)$ = The value at time t of \$1 invested at time 0.
$D'(s)/D(s) = r(s)$; $D(t) = \exp\{\int_0^t r(s)\,ds\}$

$P(t)$ = present value of \$1 received at time t.
$P(t) = \frac{1}{D(t)} = \exp\{-\int_0^t r(s)\,ds\}$

$\bar{r}(t)$ = average of the spot interest rates up to time t
$\bar{r}(t) = \frac{1}{t}\int_0^t r(s)\,ds$; $\bar{r}(t)$, $t \geq 0$ is called the yield curve.

<u>Pricing Contracts via Arbitrage</u>: this means pricing contracts so that there is no free lunch. Suppose nominal interest rate r. Stock @ \$100 / share.



At time $t=0$ can buy call option to buy 1 stock at 105.

Value of portfolio at $t=0$ of x stocks and y options is $100x + Cy$

Value of portfolio at $t=1$ is $\begin{cases} 110x + 5y & \text{if } S(1) = 110 \\ 90x & \text{if } S(1) = 90 \end{cases}$

Now, choose x and y so that value of portfolio is independent of stock price @ time 1: $110x + 5y = 90x \Rightarrow y = -4x$

GAIN @ time 1: $90x - (110x + C(-4x))(1+r)$

GAIN @ time 1 (in P.V): $90x(1+r)^{-1} - 110x + 4xC = x(4C + 90(1+r)^{-1} - 100)$

In order for there not to be arbitrage we want GAIN $= 0$ $\Rightarrow$ (since $x \neq 0$) (*)

(*) $= C = (100 - 90(1+r)^{-1})/4$. If $C \neq$ to the quantity, then there is arbitrage

<u>LAW OF ONE PRICE</u>: Two investments costs C_1 and C_2 have payouts P_1 and P_2 and $P_1 = P_2$. Then, either $C_1 = C_2$ or there is arbitrage.

<u>Pf</u>: If $C_1 < C_2$ but $P_1 \geq P_2$, then just buy investment #1 and sell #2.

<u>Example of no-arbitrage Pricing</u>:

<u>Prop</u>: For an American call option you never want to exercise it @ $t < T$.

<u>Pf</u>: If you bought one option @ $t=0$ and exercised at time $t<T$, then you received $S(t) - K$ (if $K > S(t)$, you did not excercise the option).

<u>But</u> you could have shorted the stock @ time t and paid for it at time T, paying $\min(K, S(T))$. This yields $S(t) - \min(K, S(T))e^{-r(T-t)} > S(t) - K$.

Hence, it is always preferable to wait □

<u>Prop</u>: (European Put-Call parity)

C = price of a European Call option @ $t=0$ = $C(K,T)$ ← today's \$

P = " " " " Put " " " = $P(K,T)$

$S = S(0)$ = price of stock @ time 0.

Then: either $S + P - C = Ke^{-rT}$ or there is an arbitrage opportunity.

<u>Pf</u>: Suppose $S + P - C < Ke^{-rT}$. Then, buy 1 stock, 1 put option and <u>sell</u> 1 call option @ time $t=0$. The cost of the portfolio is $S + P - C$. Now, @ time T we exercise the the put if $S(t) \leq K$ <u>OR</u> if $S(t) > K$, buyer exercises call.

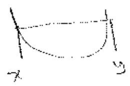
Either way we get K. But $K > (S+P-C)e^{rT}$, which is the money owe to bank.

Forward Contract: I am obligated to pay \$F @ future time T for an asset (stock) given to me @ time T.

Prop: Suppose continuous comp. interest r. Then: $F = Se^{rT}$, where $S = S(0)$.

Pf: If $F < Se^{rT}$, sell stock and invest the received amount to get Se^{rT} @ time T.
Also, buy one forward contract for delivery of one stock at time T.
@ T we have Se^{rT}, use F of this to obtain one share of the stock.
Profit $Se^{rT} - F > 0$ since $F < Se^{rT}$.

Def: A function $f: \mathbb{R} \to \mathbb{R}$ is convex if $\forall x, y \in \mathbb{R}$ and $0 \le \lambda \le 1$, we have



$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$$

Prop: Let $C(K,T)$ be the cost of a call option with strike K @ T.
a) $C(K,T)$ is convex in K and non-increasing (for fixed t).
b) for $s > 0$ $C(K,T) - C(K+s, T) \le s\, e^{-rT}$

We have an experiment with possible outcomes $\in \{1, \dots, m\}$.
e.g. how many options we buy.
Place n bets wager value x_i, $i = 1, \dots, n$
return on the ith bet if outcome of experiment is j is $r_i(j)$
Total return is $\sum_{i=1}^{n} x_i r_i(j)$ if outcome is j.
As a matrix: $R = [r_i(j)]$ (row i, col. j) $= \begin{pmatrix} r_1(1) & r_1(2) & \cdots & r_1(n) \\ r_2(1) & \cdots & & \vdots \\ r_n(1) & \cdots & & r_n(n) \end{pmatrix}$. Hence, total return $= \vec{x} \cdot R$, $\vec{x} = (x_1, \dots, x_n)$
the return if j comes up $= (\vec{x} \cdot R)_j$ = jth entry $= \vec{x} \cdot (\text{col } j \text{ of } R)$

Thm: (Arbitrage theorem) Exactly one of the following two occurs: either
a) there exists a probability vector $\vec{p} = (p_1, \dots, p_n)$, $p_j \ge 0$, $\sum p_j = 1$, s.t.
$\sum_{j=1}^{m} p_i r_i(j) = 0$, $\forall i = 1, \dots, n$ (i.e., $R \cdot \vec{p} = 0$) where $\vec{p}$ is a column vector.

OR

b) there exists a betting strategy $\vec{x} = (x_1, \dots, x_n)$ s.t.
$\sum_{i=1}^{n} x_i r_i(j) > 0$, $\forall j = 1, \dots, m$ (i.e., $\vec{x} \cdot R > 0$) every entry greater than zero

In other words: either there is a probability vector on the outcomes of the experiment that results in all bets being fair, or else there is a betting scheme that guarantees a win.

Definition: If we are in case a) then we call the probabilities $\vec{p}$ a set of risk-neutral probabilities

Pf: the proof is by linear programming, using the dual.

outcomes: {200, 50}

Example use of arbitrage theorem: Option pricing. (wagers: buy(sell) stock, buy (sell) option)

$$\begin{tikzcd} & S(1)=200 \\ S(0)=100 \arrow[ur] \arrow[dr] & \\ & 50 \end{tikzcd}$$

, C = cost of option with $K = 150$. Find C with no arbitrage.

$$\text{P.V. return from one stock} = \begin{cases} 200(1+r)^{-1} - 100 & \text{if } S(1)=200 \text{ with prob. } p \\ 50(1+r)^{-1} - 100 & \text{if } S(1)=50 \text{ with prob. } 1-p. \end{cases}$$

$$E\left[\begin{smallmatrix}\text{P.V. return} \\ \text{from one stock}\end{smallmatrix}\right] = p\left[200(1+r)^{-1}-100\right] + (1-p)\left[50(1+r)^{-1}-100\right] \overset{\text{no arbitrage}}{=} 0 \Rightarrow p = \frac{1+2r}{3}$$

the only risk-neutral probability is $(p, 1-p) = \left(\frac{1+2r}{3}, 1 - \frac{1+2r}{3}\right)$.

$$\text{P.V. return from one call} = \begin{cases} 50(1+r)^{-1} - C & \text{if } S(1)=200 \text{ with prob. } p = \frac{1+2r}{3} \\ -C & \text{if } S(1)=50 \text{ with prob } 1-p. \end{cases}$$

$$E\left[\begin{smallmatrix}\text{P.V. return} \\ \text{from one call}\end{smallmatrix}\right] = p\left[50(1+r)^{-1}-C\right] + (1-p)[-C] = \frac{1+2r}{3}\,50(1+r)^{-1} - C \overset{\text{no arbitrage}}{=} 0 \Rightarrow C = \frac{50+100r}{3(1+r)}.$$

Multiperiod binomial model

$$S(i) \begin{cases} \nearrow u\cdot S(i) = S(i+1) \\ \searrow d\cdot S(i) = S(i+1) \end{cases}$$

$0 < d < 1+r < u$. risk neutral prob: $p = \dfrac{1+r-d}{u-d}$

Stock's price after n periods $S(n) = u^Y d^{n-Y} S(0)$, where $Y = \sum_{i=1}^{n} X_i$, $X_i = \begin{cases} 1 & \text{if } S(i)=uS(i-1) \\ 0 & \text{if } S(i)=dS(i-1) \end{cases}$

$Y \sim Bin(n, p = \frac{1+r-d}{u-d})$. Value of option after n periods $= (S(n)-K)^+$, so, P.V. of owning option is: $(1+r)^{-n}(S(n)-K)^+$, the expectation is: $E[(1+r)^{-n}(S(n)-K)^+] = (1+r)^{-n}E[(S(n)-K)^+]$

So, the cost C that does not result in arbitrage is: $C = (1+r)^{-n}E[(S(0)u^Y d^{n-Y} - K)^+]$.

<u>Note</u>: If we assume that the underlying security follows a Geometric B.M, then this formula becomes Black-Scholes, in other words

$C = e^{-rt} E[(S(0)e^W - K)^+]$, where $W \sim Normal((r - \sigma^2/2)t, \sigma^2 t)$ is the unique no-arbitrage cost of a call option to purchase the security at time t for the specified price K. <u>Note</u>: under the risk-neutral G.B.M $S(t)|S(0)$ is a lognormal random variable with mean $= (r - \frac{\sigma^2}{2})t$ and var $= \sigma^2 t$

$\hookrightarrow$ explicitely evaluate to obtain Black-Scholes:

$$\boxed{C = S(0)\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}), \text{ where, } \omega = \frac{rt + \sigma^2 t/2 - \log(K/S(0))}{\sigma\sqrt{t}}}$$