CHAPTER 8:

Options with dividends: (3 different models).

Model 1: the dividend on $S(t)$ is distributed continuously as a fraction f of $S(t)$.

Value of this investment @ t $\Leftrightarrow$ $\boxed{M(t) = e^{ft} S(t)}$ Assume $M(t) \sim$ G.B.M.

$M(t) = M(0) e^{W(t)}$, $W(t) \sim \text{Normal}\left(\left(r - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$.

Then, $\frac{S(t)}{S(0)} = \frac{e^{-ft} M(t)}{M(0)} = e^{-ft} e^{W(t)} \Rightarrow \boxed{S(t) = S(0) e^{-ft} e^{W(t)}}$ under risk-neutral probabilities.

By Arbitrage theorem, let C_1 = call price w/ div. type 1, expiry T strike K.

Then, $C_1 = e^{-rT} E[\text{return @ time } T] = e^{-rT} E[(S(t)-K)^+] = e^{-rT}[(S(0)e^{-fT}e^{W(T)} - K)^+]$

By B.S $\boxed{C_1 = C(S(0)e^{-fT}, K, T, r, \sigma)}$ (same price as if there were no dividends, but the initial price were $S(0)e^{-fT}$).

Model 2: At time t_d you get a distribution of $fS(t_d)$. The price of the share decreases: $S(t_d^+) = S(t_d^-) - fS(t_d^-) = (1-f)S(t_d^-) \Rightarrow S(t_d^-) = \frac{1}{1-f} S(t_d^+)$.

We can reinvest (buy stocks) for $\frac{f}{1-f} S(t_d^+)$. The market value @ time t:

$M(t) = \begin{cases} S(t) & t < t_d \\ \frac{1}{1-f} S(t) & t \geq t_d \end{cases}$. Suppose $M(t) \sim$ G.B.M, volatility parameter σ.

The risk-neutral probabilities for this are that of G.B.M with σ and $r - \frac{\sigma^2}{2}$.

So, if $t < t_d$ then the call is just B-S: $C_2 = C(S(0), K, T, r, \sigma)$.

if $t \geq t_d$ then $\frac{S(t)}{S(0)} = (1-f)\frac{M(t)}{M(0)} = (1-f)e^{W(t)} \Rightarrow \boxed{S(t) = S(0)(1-f)e^{W(t)}}$ under risk-neutral prob.

Thus, $C_2 = e^{-rT} E[\text{return @ time } T] = e^{-rT} E[(S(t)-K)^+] = e^{-rT}[(S(0)(1-f)e^{W(t)}]$

By B.S. $\boxed{C_2 = C(S(0)(1-f), K, T, r, \sigma)}$

Model 3: At t_d get distribution of D \$/share. Since for $t<t_d$ $S(t) > De^{-r(t_d - t)}$ (or else there is arbitrage) $S(t)$ is not G.B.M b/c of deterministic part.

$S(t) = S^*(t) + De^{-r(t_d - t)}$, for $t < t_d$. Assume $S^*(t) \sim$ G.B.M. Then,

$S^*(t) = S^*(0)e^{W(t)}$ $\Rightarrow$ $S^*(0) = S(0) - De^{-rt_d}$

$C_3 = e^{-rT} E[\text{return @ time } T] = e^{-rT} E[(S(T)-K)^+] = e^{-rT} E[(S^*(T) + De^{-r(t_d - T)} - K)^+]$

$= e^{-rT} E[(\underbrace{(S(0) - De^{-rt_d})}_{\text{New } S(0)} e^{W(T)} - \underbrace{(K - De^{-r(t_d - T)})}_{\text{new } K})^+]$

$\boxed{C_3 = C(S(0) - De^{-rt_d}, K - De^{-r(t_d - T)}, T, r, \sigma)}$ for $T < t_d$

If $T > t_d$ then the stock's price suddenly drops by D at t_d:
$S(t) = S^*(t)$ if $t > t_d$. $C_3 = P.V.\ E[\text{retur @ } T] = e^{-rT} E[(S^*(T)-K)^+]$
$= e^{-rT} E[(S^*(0)e^{W(T)} - K)^+] \Rightarrow \boxed{C_3 = C(S(0) - De^{-rt_d}, K, T, r, \sigma)}$

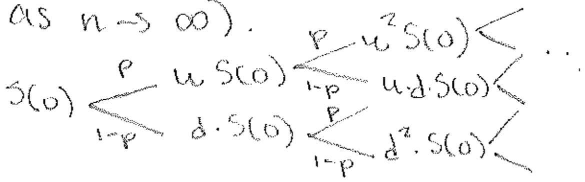
Pricing an American Put (can exercise at any $t < T$, V = price of put).

the no arbitrage price should be the P.V. expected return from an optimal excersice strategy, assume risk-neutral G.B.M for $S(t)$.

We approximate this using time intervals of size $\delta = T/n$. Let $t_i = \frac{i}{n}T = i\delta$.

$$S(t_{i+1}) = \begin{cases} u \cdot S(t_i) & \text{prob. } p \\ d \cdot S(t_i) & \text{prob } 1-p \end{cases}, \text{ where } p = \frac{1 + r\delta - d}{u - d},\ u = e^{\sigma\sqrt{\delta}},\ d = e^{-\sigma\sqrt{\delta}}.$$

(we know that this discrete approx. becomes the risk-neutral G.B.M as $n \to \infty$).



$S(t_k) = u^i d^{k-i} S(0)$, $i = 0, \dots, k$
with probability $\binom{k}{i} p^i (1-p)^{k-i}$

Let $V_k(i) = E[\text{return@ time } t_k \text{ of the } i^{th} \text{ branch}]$ (assuming: $S(t_k) = u^i d^{k-i} S(0)$ and that we didn't previously exercise optimal strategy for exercising).

To determine $V_0(0)$, first use (8.0) to determine the values of $V_n(i)$, then use (8.1) with $k = n-1$ to obtain $V_{n-1}(i)$, then use (8.1) again for $V_{n-2}(i)$...

$$\boxed{\begin{aligned} &V_n(i) = \max(K - u^i d^{n-i} S(0), 0), \quad i = 0, \dots, n. \ (8.0) \\ &V_k(i) = \max(K - u^i d^{k-i} S(0), e^{-r\delta}(p V_{k+1}(i+1) + (1-p)V_{k+1}(i))) \quad i = 0, \dots, k.\ (8.1) \end{aligned}}$$

Geometric Brownian Motion with Jumps:

$S(t) = S^*(t) \prod_{i=1}^{N(t)} J_i$, where $N(t)$ = # of jumps in time interval $[0, t]$

Let $J(t) = \prod_{i=1}^{N(t)} J_i$. We define $J(t) = 1$ if $N(t) = 0$.

$S^*(t), t \geq 0$ is G.B.M. $J_i > 0$ are the jumps. $J_i > 1$ jump up, $J_i < 1$ jumpdown.

$N(t)$ follows a Poisson Process:
- $N(0) = 0$ and # events in any two disjoint intervals are independent
- the # of events in an interval only depends on its length

$P(N(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$ (parameter λt).

J_i are independent random variables but with some distribution (i.i.d).
Say J_i follow J_0 distrib.

$$E[J(t)] = e^{-\lambda t(1-E[J_0])} \quad ; \quad Var[J(t)] = e^{\lambda t(1-E[J_0^2])} - e^{-2\lambda t(1-E[J_0])}$$

Hence,

$$E[S^*(t)] = S(0)\, e^{(\mu+\frac{\sigma^2}{2})t},$$

by independence of $S^*(t)$ AND $J(t)$

$$E[S(t)] = E[S^*(t)J(t)] = E[S^*(t)]E[J(t)] = S(0)e^{(\mu+\frac{\sigma^2}{2})t} \cdot e^{-\lambda t(1-E[J_0])}$$

$$\boxed{E[S(t)] = S(0)\, e^{(\mu+\frac{\sigma^2}{2})t - \lambda t(1-E[J_0])}}$$

Risk-neutral probability imply $E[S(t)]e^{-rt} = S(0) \Rightarrow S(0)e^{rt} = S(0)e^{(\mu+\frac{\sigma^2}{2})t - \lambda t(1-E[J_0])}$

$\Rightarrow r = \mu + \frac{\sigma^2}{2} - \lambda + \lambda E[J_0] \Rightarrow \boxed{\mu = r - \frac{\sigma^2}{2} + \lambda - \lambda E[J_0]}$ risk-neutral prob. for $S(t)$ result when using this μ. for GBM $S(t)$.

No Arbitrage $\Rightarrow E[\text{return of a call} - C_J] = 0$, where C_J = price of call with jumps instead.

$\Rightarrow C_J = E[\text{return of call}] = E[(S^*(T)J(T) - K)^+ e^{-rT}]$

$\Rightarrow \boxed{C_J = e^{-rT} E[(J(T)S(0)e^{W(T)} - K)^+]}$, $S(0)$ = initial price of stock.

$W \sim Normal\left((r - \sigma^2/2 + \lambda - \lambda E[J_0])t,\ t\sigma^2\right)$.

Assume J_0 is log normal: $J_0 = e^{NRV(\mu_0, \sigma_0)} \Rightarrow E[J_0] = e^{\mu_0 + \sigma_0^2/2}$.

Let $J_i = e^{X_i} \Rightarrow J(t) = e^{\sum_{i=1}^{N(t)} X_i}$.

$$C_J = e^{-rT} E\left[\left(S(0)\, e^{W_T + \sum_{i=1}^{N(T)} X_i} - K\right)^+\right].$$

Suppose $N(T) = n \Rightarrow E[W_T + \sum_{i=1}^{N(T)} X_i \mid N(T) = n] = E[W_T + \sum_{i=1}^{n} X_i] = \underbrace{(r + \frac{\sigma^2}{2} + \lambda - \lambda E[J_0])t + n\mu_0}$ mean W_T; sum of ind. normal is normal

If $N(T) = n$ then $W_T + \sum_{i=1}^{n} X_i$ has mean $(r(n) - \frac{\sigma^2(n)}{2})T$ and variance $\sigma^2(n)T$.

where $r(n) = r + \lambda - \lambda E[J_0] + \frac{n}{t}\log E[J_0]$.

$$e^{-r(n)T} E\left[(S(0)e^{W_T + \Sigma X_i} - K)^+ \mid N(T) = n\right] = C(S(0), K, T, r(n), \sigma(n)).$$

$$C_J = e^{-rT} E\left[(S(0)e^{W_T + \Sigma X_i} - K)^+\right] = e^{-rT}\sum_{n=0}^{\infty} E\left[S_0 e^{W_T + \Sigma X_i} \mid N(t) = n\right] P(N(t) = n)$$

$$\boxed{C_J = \sum_{n=0}^{\infty} e^{-\lambda t E[J_0]} \frac{(\lambda T E[J_0])^n}{n!} \cdot C(S(0), K, T, r(n), \sigma(n))}$$

theorem 8.4.2 Assuming a general distribution for the size of a jump, the no-arbitrage option cost $= E[C(S(T)J(T), T, K, \sigma, r)]$
Moreover, no-arbitrage option $\geq C(S(0), T, K, \sigma, r)$.
Cost $C_J \approx C(S(0), T, K, \sigma, r) + s \dots$

Estimating Volatility: In B-S equation: $S(0), T, K$ are known and r maybe variable but predictable to some degree (set up by Fed). σ has to be estimated for the future interval $[0, T]$.

let $X_1, \dots, X_n$ be independent random variables (i.i.d) with μ_0, σ_0^2.

$\boxed{\bar{X} = \frac{X_1 + \dots + X_n}{n}}$ as $n \to \infty$ (with high prob), this will tend to μ_0.
This is unbiased $E[\bar{X}] = \mu_0$

estimator of μ_0.

For variance the estimator is $\varepsilon^2 = \frac{n}{n-1} S_1 = \boxed{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \varepsilon^2}$

This is an unbiased estimator: $E[\varepsilon^2] = \sigma_0^2$

For N.R.V. $Var(\varepsilon^2) = \frac{2\sigma_0^4}{n-1}$

MEAN SQUARE ERROR $\boxed{MSE = E[(\varepsilon^2 - \sigma_0^2)^2] = Var(\varepsilon^2)}$

More fancy estimators

1) Assume stock price follow G.B.M. let $\delta = \frac{T}{n}$ be time periods. Suppose we know $S(i \cdot \delta)$, $i = 1, \dots, n$ (so $S(0)$ is now the past).

$$X_1 = \log \frac{S(\delta)}{S(0)}, \quad X_2 = \log \frac{S(2\delta)}{S(\delta)}, \dots, \quad X_n = \log \frac{S(n\delta)}{S((n-1)\delta)}.$$

Under G.B.M Assumption, $X_i \sim Normal(\delta\mu, \delta\sigma^2)$.

$$\sum_{i=1}^{n} X_i = \log \frac{S(\delta)}{S(0)} + \log \frac{S(2\delta)}{S(\delta)} + \dots + \log \frac{S(n\delta)}{S((n-1)\delta)} = \log \left[\frac{S(\delta)}{S(0)} \cdot \frac{S(2\delta)}{S(\delta)} \cdots \frac{S(n\delta)}{S((n-1)\delta)} \right] = \log \frac{S(n\delta)}{S(0)}$$

$\log \frac{S(T)}{S(0)}$ ''

So, $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{1}{n} \log \frac{S(T)}{S(0)} \Rightarrow \boxed{\bar{X} = \frac{1}{n} \{ \log(S(T)) - \log(S(0)) \}}$ Typically close to 0.

$\hat{\sigma}^2 = \frac{1}{\delta(n-1)} \sum_{i=1}^{n} (X_i - \bar{X})^2$, using fact that this are normal $Var(\hat{\sigma}^2) = \frac{2\sigma^4}{n-1}$

The advise is to choose $\delta = 1/252$ (252 days in US trading year)

take $S(i\delta)$ to be closing price on day i.

2) Using Opening & Closing DATA O_i = opening price on day i ; C_i = closing price

$\log \frac{C_i}{C_{i-1}} = \log \frac{C_i}{O_i} \cdot \frac{O_i}{C_{i-1}} = \log \frac{C_i}{O_i} + \log \frac{O_i}{C_{i-1}}$. Assuming C_i/O_i indep. of O_i/C_{i-1}

$Var\left(\log \frac{C_i}{C_{i-1}}\right) = Var\left(\log \frac{C_i}{O_i}\right) + Var\left(\log \frac{O_i}{C_{i-1}}\right)$. So, get new estimator:

$$\hat{\sigma}_1^2 = \frac{252}{n-1} \sum_{i=1}^{n} \left(\log C_i - \log O_i - \cancel{\bar{X}}^{\,0}\right)^2 + \left(\log O_i - \log C_{i-1}\right)^2$$

3) Using Opening, Closing and High-Low DATA

(Read on book)

Chapter 9: No Arbitrage Pricing can lead to Multiple prices.

Use utility to distinguish between probabilities vectors.

$E[u(X)] \geq E[u(Y)] \implies$ choose investment X.

The utility function is specific to an investor. A common Assumption (Law of diminishing returns) is that $u(x)$ is a nondecreasing function of x. Moreover, for fixed $\Delta > 0$, $u(x+\Delta) - u(x)$ is nonincreasing in x. $\iff$ concave. In other words, $u'(x)$ should be decreasing, $u''(x) \leq 0$.

Jensen's Inequality $\boxed{u \text{ concave: } E[u(X)] \leq u(E[X])}$ this is mathematics for saying that An investor with a concave utility function is risk-averse.

Let X = return from an investment. Jensen's Inequality states that the investor would prefer the certain return $E[X]$ to receiving a random return with this mean.

A MATHEMATICAL convenient utility function is $\boxed{u(x) = \log(x)}$ it is concave! So an investor with a log utility is risk-averse.

Let W_0 be your initial wealth. After 1 period $W_1 = X_1 W_0$, X_1 is a R.V. So, after n periods $W_n = X_n X_{n-1} \cdots X_2 X_1 W_0$, where X_i has a specific distrib.

Let R_n = rate of return. Then, $W_n = (1+R_n)^n W_0 \implies (1+R_n)^n = \frac{W_n}{W_0} = X_n \ldots X_1$.

$\implies \log((1+R_n)^n) = \log(X_1 \cdots X_n) \implies \boxed{\log(1+R_n) = \frac{1}{n}\sum_{i=1}^{n} \log(X_i)}$ Since X_i are i.i.d,

Strong law of large number $\implies \log(1+R_n) = \frac{1}{n}\sum \log(X_i) \xrightarrow{n\to\infty} E[\log(X)]$.

this means that the long-run rate of return is maximized by choosing the investment that yields the largest value of $\underline{E[\log(X)]}$

Moreover, because $W_n = W_0 X_1 \cdots X_n \Rightarrow \log(W_n) = \log(W_0) + \sum_{i=1}^{n} \log(X_i)$

$\Rightarrow E[\log(W_n)] = \log(W_0) + n\,E[\log(X)]$. So, maximizing $E[\log(X)]$ is equivalent to maximizing the expectation of the log of the final wealth.

<u>Portfolio SELECTION</u>: w_i is invested in each security $i = 1, \dots, n$. the end-of-period (1 period) wealth is $W = \sum_{i=1}^{N} w_i X_i$, where X_i is a non-negative random variable.

the vector $w_1, \dots, w_n$ is called <u>portfolio</u>. the main problem here is, given a utility funtion $u(x)$, determine the portfolio that maximizes the expected utility of one's end-of-period wealth:

choose $w_1, \dots, w_n$, s.t $w_i \geq 0$, $i = 1, \dots, n$, $\sum_{i=1}^{n} w_i = w$ ← positive amount to be invested. To Maximize $E[u(W)]$

<u>Assumption</u>: <u>W is normal</u>. this makes sense if n is large, by CLT. (and i.i.d)

Now suppose: $u(x) = 1 - e^{-bx}$ (exp. utility $b>0$). then,

$$E[u(W)] = E[1 - e^{-bW}] = 1 - E[e^{-bW}] = 1 - e^{-bE[W] + \frac{b^2}{2}Var(W)} = 1 - e^{-b\{E[W] - \frac{b}{2}Var(W)\}}$$

So, to maximize $E[u(W)]$, it suffices to max. $\boxed{E[W] - \frac{b}{2} Var(W)}$

Note: if two portfolios have random end-of-periods wealths W_1 and W_2 such that $E[W_1] \geq E[W_2]$ and $Var(W_1) \leq Var(W_2) \Rightarrow E[U(W_1)] \geq E[U(W_2)]$ (makes sense)

<u>MEAN & VARIANCE OF W for a given portfolio</u>:

The rate of return for i's security: $R_i = X_i - 1$. $\Rightarrow r_i = E[R_i]$, $v_i^2 = Var(R_i)$

$$\Rightarrow W = \sum_{i=1}^{n} w_i(1 + R_i) = W_0 + \sum_{i=1}^{n} w_i R_i \Rightarrow E[W] = W_0 + \sum_{i=1}^{n} E[w_i R_i] = W_0 + \sum_{i=1}^{n} w_i E[R_i]$$

$$Var(W) = Var\left(W_0 + \sum_{i=1}^{n} w_i R_i\right) = 0 + Var\left(\sum_{i=1}^{n} w_i R_i\right) = \sum_{i=1}^{n} Var(w_i R_i) + \sum_{i=1}^{n} \sum_{j \neq i} Cov(w_i R_i, w_j R_j)$$

$$\Rightarrow Var(W) = \sum_{i=1}^{n} w_i^2 v_i^2 + \sum_{i=1}^{n} \sum_{j \neq i} w_i w_j c(i,j), \text{ where } c(i,j) = Cov(R_i, R_j)$$

<u>For more General utilities functions</u>: Condition: u'' exists and is nondecreasing.

By expanding $u(W)$ about the point $\mu = E[W]$, taking expectation, we get: an approximation is the portfolio that maximizes $U(E[W]) + U''(E[W])\frac{Var(W)}{2}$

<u>Estimators for Covariance</u>:

$$Cov(R_i, R_j) = E[(R_i - r_i)(R_j - r_j)] \Rightarrow \frac{\sum_{k=1}^{m} (r_{i,k} - \bar{r}_i)(r_{j,k} - \bar{r}_j)}{m-1}$$

$$\bar{r}_i = \frac{\sum_{k=1}^{m} r_{i,k}}{m}, \quad \bar{r}_j = \frac{\sum_{k=1}^{m} r_{j,k}}{m}$$

CAPM: $\beta_i = \dfrac{(R_i - r)}{(R_M - r)}$, R_i = R.o.R. investment # i; r = risk free interest rate
R_M = R.o.R. for a whole market (index fund)

Also, $\beta_i = \dfrac{cov(R_i, R_M)}{Var(R_M)}$

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CHAPTER 10: Stochastic Order Relations.

Def: $X \geq_{st} Y$ if $\forall t \in Dom(X) = Dom(Y) = \mathbb{R}$: $P(X > t) \geq P(Y > t)$

This definition implies $F_X(t) \leq F_Y(t)$ $\forall t$, where F is the cumulative dist.

Proposition: $X \geq_{st} Y \iff E[h(X)] \geq E[h(Y)]$ for all increasing functions h.

Using Coupling: to show $X \geq_{st} Y$ it is enough to find $X' > Y'$ s.t. X and X', and Y and Y' share the same PDF. (X' and Y' are the coupled variables to X and Y)

theorem: If $X \geq_{s.t} Y$ then $\exists$ X' & Y' with same PDF as X & Y s.t. $X' \geq Y'$.

theorem: Let $(X_1, .., X_N)$ and $(Y_1, .., Y_N)$ be vectors of independent R.V's st $X_i \geq_{s.t} Y_i$. Then, for any increasing multivariable $g: \mathbb{R}^N \to \mathbb{R}$, we have $g(X_1, .., X_N) \geq_{s.t.} g(Y_1, .., Y_N)$

Likelihood Ratio Ordering

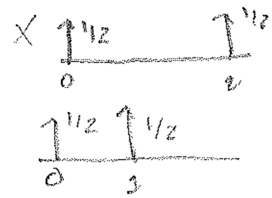
Def: $X \geq_{\ell r} Y$ if $f_X(t)/f_Y(t)$ is nondecreasing for all t! region where $f(x)$ or $g(x)$ is greater than zero.

(in case X, Y are continuous: $P(X = x)/P(Y = x)$ is nondecreasing in x)

this is over the region where either $P(X = x)$ or $P(Y = x)$ is greater than 0.

Proposition. $X \geq_{\ell r} Y \Rightarrow X \geq_{s.t} Y$.

Note that $X \geq_{s.t} Y \not\Rightarrow X \geq_{\ell r} Y$. e.g



SECOND ORDER DOMINANCE

Def: $X \geq_{icv} Y \iff E[h(X)] \geq E[h(Y)]$ for all increasing concave functions h

Thm: $X \geq_{icv} Y \iff \int_{-\infty}^{a} P(X < s)\,ds \leq \int_{-\infty}^{a} P(Y < s)\,ds$