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Moments provide a way of describing distributions different from density functions or c.d.f.s, and better suited to the CLT.

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If $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$, then

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In other words we can compute the expected value of any polynomial function of X just using the M_n's.

We can even compute things like

$$\mathrm{P}(a \leq X \leq a + \mathrm{d}x).$$

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$$b(x) = \frac{1}{\sqrt{2\pi\epsilon}} \mathrm{e}^{-\frac{1}{2}\left(\frac{x-a}{\epsilon}\right)^2}.$$

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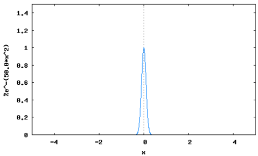
And the shape of b is useful because it focuses attention on a small range of x values.



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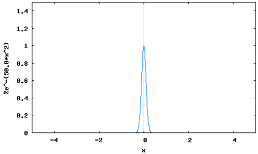
If ϵ is small enough, then we can assume that f is constant over almost the entire bell.



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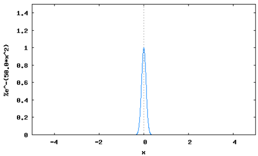

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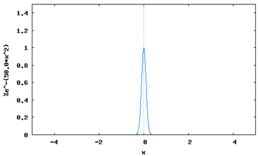

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$$\approx 0 + f(a) \int_{|X-a|\leq 3\epsilon} b(x)\,\mathrm{d}x \approx f(a).$$

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And that in turn means that f is determined by the M_n's, because although $b(x)$ is not a polynomial, it is a *limit* of polynomials:

$$b(x) = \frac{1}{\sqrt{2\pi\epsilon}} \mathrm{e}^{-\frac{1}{2}(\frac{x-a}{\epsilon})^2}$$

$$= \frac{1}{\sqrt{2\pi\epsilon}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}(\frac{x-a}{\epsilon})^2)^k}{k!}$$

In a similar way it can be shown that those functions b also determine the probabilities in discrete distributions.

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I'll consider it established now that the M_n's determine all the properties of a distribution. Hence if we can show that two distributions have finite and equal M_n for all n we will know they are actually the same distribution.
Now let's use the moments M_n to prove the Central Limit Theorem.

Central Limit Theorem

Theorem

Let X_i for $i = 1, 2, 3, \ldots$ be i.i.d. RVs with mean 0 and s.d. 1. Define

$$A_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i.$$

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Then as $n \to \infty$ *the distribution of* A_n *approaches the standard normal.*

Note that the particular distribution of the X_i's *does not matter* — the limiting distribution is the same no matter what you start out with.

The basic idea is to show that the moments of the limiting distribution depend only on the first and second moments of the X_i — nothing else.

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Let's warm up by computing a few easy moments of A_n.

$$\mathrm{E}(A_n) = \mathrm{E}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right) = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}\mathrm{E}(X_i)\right) = 0$$

because all the X_i have mean 0.

Second moment:

$$\mathrm{E}(A_n^2) = \mathrm{E}\left(\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right)^2\right) = \frac{1}{n}\left[\sum_{i=1}^{n}\mathrm{E}(X_i^2) + \sum_{i\neq j}\mathrm{E}(X_i X_j)\right]$$

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So now we know that A_n has mean 0 and s.d. 1 for all n.

We conclude therefore the limiting distribution of the A_n also has mean 0 and s.d. 1.

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But this expression goes to 0 as $n \to \infty$.
So the third moment of the limiting distribution of A_n is 0.

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There's another pattern lurking here though...

Consider an example: Suppose we're computing the fifth moment of A_n. One expression we'll have to deal with is

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Multiplying these two expressions together gives $\frac{M_2 M_3}{\sqrt{n}}$ which once again approaches 0 as $n \to \infty$.

Evidently very few terms survive the limit. In fact the only expressions that don't cancel in the limit are those involving only second moments of X_i:

$$\frac{1}{n^{k/2}} \sum_{\text{distinct } i_1, i_2, \ldots, i_{k/2}} \mathrm{E}(X_{i_1}^2)\mathrm{E}(X_{i_2}^2) \ldots \mathrm{E}(X_{i_{k/2}}^2)$$

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so we conclude that odd moments of A_n approach 0 in the limit.

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I.e., if the X_i's are standard normal then so are the A_n's.

And if every single A_n is standard normal, then the limiting distribution is also.

To put this together:
Since there is *some* distribution for the X_i which produces the standard normal as the limit of A_n,

then in fact *every* distribution, once it's been standardized to have mean 0 and s.d. 1, must produce the standard normal in the limit also.

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