Conditional Densities: Example

Let X and Y be RVs on the unit square $[0,1]\times[0,1]$ with joint density function

$$f_{X,Y}(x,y) = x + \frac{3}{2}y^2.$$

Then the marginal densities are

$$f_Y(y) = \frac{3\,y^2+1}{2}$$
$$f_X(x) = \frac{2\,x+1}{2}.$$

From these we get

$$\mu_X = \mathrm{E}(X) = \frac{7}{12} \qquad \mu_Y = \mathrm{E}(Y) = \frac{5}{8}$$
$$\sigma_X = \mathrm{SD}(X) = \frac{\sqrt{11}}{12} \qquad \sigma_Y = \mathrm{SD}(Y) = \frac{\sqrt{73}}{8\sqrt{15}}.$$

To calculate the correlation coefficient, first get

$$\mathrm{E}(XY) = \int_0^1\int_0^1 xy(x+\frac{3}{2}y^2)\,\mathrm{d}x\,\mathrm{d}y = \frac{17}{48},$$

giving

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mu_X\mu_Y = -\frac{1}{96}$$
$$\rho = \mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} = -\frac{\sqrt{15}}{\sqrt{11}\,\sqrt{73}} = -.13667,$$

indicating a weak negative correlation.

The best linear approximation to Y in terms of X is $Y^* \approx \rho X^*$ giving

$$Y \approx \frac{5}{8} - \frac{\sqrt{73}}{8\sqrt{15}}(.13667)\left(\frac{X-\frac{7}{12}}{\frac{\sqrt{11}}{12}}\right) = -.1364x + .7045.$$

The best approximation in terms of *any* function of X is $Y \approx \mathrm{E}(Y|X)$. For this we need the conditional distribution $Y|X$, for which the density function is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{3y^2+2x}{2x+1}.$$

From this we get

$$\mathrm{E}(Y|X) = \int_0^1 y\frac{3y^2+2X}{2X+1}\,\mathrm{d}y = \frac{4X+3}{4(2X+1)}.$$

We can also use $f_{Y|X}$ to calculate the conditional variance of Y.

$$\mathrm{E}(Y^2|X) = \int_0^1 y^2\frac{3y^2+2X}{2X+1}\,\mathrm{d}y = \frac{10X+9}{15(2X+1)}$$

$$\mathrm{Var}(Y|X) = \mathrm{E}(Y^2|X) - \mathrm{E}(Y|X)^2 = \frac{80X^2+88X+9}{240(2X+1)^2}.$$

At least that's what Maxima tells me.