M464 - Introduction To Probability II - Homework 1

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Chapter 3

(1.1) A simplified model for the spread of a disease goes this way: The total population size is $N = 5$, of which some are diseased and the remainder are healthy. During any single period of time, two people are selected at random from the population and assumed to interact. The selection is such that an encounter between any pair of individuals in the population is just as likely as between any other pair. If one of these persons is diseased and the other not, then with probability $\alpha = 0.1$ the disease is transmitted to the healthy person. Otherwise, no disease transmission takes place. Let X_n denote the number of diseased persons in the population at the end of the nth period. Specify the transition probability matrix.

Solution: First note that $P_{0,0} = P_{5,5} = 1$, i.e., the probability of having no diseased persons if you have no diseased person is 1 (there is no way to transmit disease), and the probability of having all 5 persons diseased given that they are all diseased is also 1. Moreover, the probability of incrementing the number of diseased persons by two or more than the current diseased number is zero. In other words, there can only be increments of one more disease person or no increments at all. This information already gives us the following transition probability matrix:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ \left\| \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & 0 & & & 0 & 0 \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

Therefore, we need only compute the probabilities $P_{i,i+1}$ and $P_{i,i}$ for $i = 1, 2, 3, 4$. Note that this probabilities follow a Hypergeometric distribution: Let i denote the number of diseased people. Then,

$$P_{i,i+1} = 0.1 \frac{\binom{i}{1}\binom{5-i}{1}}{\binom{5}{2}} = 0.1 \frac{i(5-i)}{\frac{5!}{3!2!}} = \frac{i(5-i)}{100}, \text{ for } i = 1, 2, 3, 4$$

where 0.1 is the probability of disease being transmitted to a healthy person and the rest of the term is the probability of selecting exactly one disease person and one healthy person out of all possible ways of making the selection. The complement of this probability would be the probability of having no transmition of disease, i.e.:

$$P_{i,i} = 1 - P_{i,i+1} = 1 - \frac{i(5-i)}{100} = \frac{100 - i(5-i)}{100}, \text{ for } i = 1, 2, 3, 4$$

Hence, the transition probability matrix is:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ \left\| \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.96 & 0.04 & 0 & 0 & 0 \\ 0 & 0 & 0.06 & 0.94 & 0 & 0 \\ 0 & 0 & 0 & 0.06 & 0.94 & 0 \\ 0 & 0 & 0 & 0 & 0.04 & 0.96 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

(1.4) The random variables $\xi_1, \xi_2, \ldots$ are independent and with the common probability mass function

$k =$	0	1	2	3
$Pr\{\xi = k\} =$	0.1	0.3	0.2	0.4

Set $X_0 = 0$, and let $X_n = max\{\xi_1, \ldots, \xi_n\}$ be the largest ξ observed to date. Determine the transition probability matrix for the Markov chain $\{X_n\}$.

Solution:

We wish to compute $P_{ij} = Pr\{X_{n+1} = j|X_n = i\}$ for $i, j \in \{0, 1, 2, 3\}$. We know that $Pr\{X_0 = 0\} = 1$. Now, $Pr\{X_1 = k|X_0 = 0\} = Pr\{X_1 = k\} = Pr\{max\{\xi_1\} = k\} = Pr\{\xi_1 = k\}$, and we are given this distribution. Therefore, the transition from state 0 to any other state $k = 0, 1, 2, 3, 4$ is the same as the distribution given, $0.1, 0.3, 0.2, 0.4$ respectively.
Now, if the Chain is in state 1, we know that $Pr\{X_{n+1} = 0|X_n = 1\} = Pr\{max\{\xi_1, \ldots, 1, 0\} = 0\} = 0$, i.e., we already observed a value greater than 0 and so there is no possible transition to state 0. Likewise, we get that $Pr\{X_{n+1} = j|X_n = i\} = 0$ if $j < i$.
Next,

$$\begin{aligned}
Pr\{X_{n+1} = 1|X_n = 1\} &= Pr\{max\{\xi_1, \ldots, 1, \xi_{n+1}\} = 1\} && \text{by definition} \\
&= Pr\{max\{1, \xi_{n+1}\} = 1\} && \text{Markov property} \\
&= Pr\{\xi_{n+1} = 0 \text{ OR } \xi_{n+1} = 1\} = 0.1 + 0.3 = 0.4 && \text{since these are disjoint events}
\end{aligned}$$

A similar analysis shows that to transition from state 2 to state 2 we have:

$$Pr\{X_{n+1} = 2|X_n = 2\} = Pr\{\xi_{n+1} = 0 \text{ OR } \xi_{n+1} = 1 \text{ OR } \xi_{n+1} = 2\} = 0.1 + 0.3 + 0.2 = 0.6$$

Lastly, for state 3:

$$Pr\{X_{n+1} = 3|X_n = 3\} = Pr\{\xi_{n+1} = 0 \text{ OR } \xi_{n+1} = 1 \text{ OR } \xi_{n+1} = 2 \text{ OR } \xi_{n+1} = 3\} = 0.1 + 0.3 + 0.2 + 0.4 = 1$$

All other probabilities are determined by the given distribution. Hence, the transition probability matrix is:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline 0.1 & 0.3 & 0.2 & 0.4 \\ 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 1 \end{array}$$

(2.4) Suppose X_n is a two-state Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{cc} 0 & 1 \\ \hline \alpha & 1-\alpha \\ 1-\beta & \beta \end{array}$$

Then $Z_n = (X_{n-1}, X_n)$ is a Markov chain having the four states $(0, 0), (0, 1), (1, 0)$, and $(1, 1)$. Determine the transition probability matrix.

Solution: To compute this transition matrix we use the definition of Z_n as follow:

$$\begin{aligned}
P\{Z_n = (0,0)|Z_{n-1} = (0,0)\} &= \frac{P\{Z_n = (0,0), Z_{n-1} = (0,0)\}}{P\{Z_{n-1} = (0,0)\}} && \text{conditional prob.} \\
&= \frac{P\{(X_{n-1}, X_n) = (0,0), (X_{n-2}, X_{n-1}) = (0,0)\}}{P\{(X_{n-2}, X_{n-1}) = (0,0)\}} && \text{Def. of } Z_n \\
&= \frac{Pr\{X_{n-1} = 0, X_n = 0, X_{n-2} = 0\}}{P\{X_{n-2} = 0, X_{n-1} = 0\}} && \text{Ordered pairs} \\
&= \frac{Pr\{X_n = 0|X_{n-1} = 0, X_{n-2} = 0\}P\{X_{n-1} = 0, X_{n-2} = 0\}}{P\{X_{n-2} = 0, X_{n-1} = 0\}} && \text{conditional prob.} \\
&= Pr\{X_n = 0|X_{n-1} = 0, X_{n-2} = 0\} && \text{simplifying} \\
&= Pr\{X_n = 0|X_{n-1} = 0\} && \text{Markov property} \\
&= \alpha && \text{Given}
\end{aligned}$$

We can proceed in this manner to compute all of the transition probabilities (I won't type all of them here to save space). Note that exactly half of the entries in the matrix are zero. This is due to events which turn out to be impossible, e.g.: $Pr\{Z_n = (0,0)|Z_{n-1} = (1,1)\}$ computed as follow:

$$
\begin{aligned}
P\{Z_n=(0,0)|Z_{n-1}=(1,1)\} &= \frac{P\{Z_n=(0,0),Z_{n-1}=(1,1)\}}{P\{Z_{n-1}=(1,1)\}} && \text{conditional prob.}\\
&= \frac{P\{(X_{n-1},X_n)=(0,0),(X_{n-2},X_{n-1})=(1,1)\}}{P\{(X_{n-2},X_{n-1})=(1,1)\}} && \text{Def. of } Z_n\\
&= \frac{Pr\{X_{n-1}=0,X_n=0,X_{n-2}=1,X_{n-1}=1\}}{P\{X_{n-2}=1,X_{n-1}=1\}} && \text{Ordered pairs}\\
&= 0. \quad \text{due to the impossible event } X_{n-1}=0 \text{ and } X_{n-1}=1
\end{aligned}
$$

Hence, the transition probability matrix is:

$$
\mathbf{P} = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array}
\begin{array}{cccc} (0,0) & (0,1) & (1,0) & (1,1) \\ \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & 1-\beta & \beta \\ \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & 1-\beta & \beta \end{array}
$$

(2.5) A Markov chain has the transition probability matrix:

$$
\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{ccc} 0 & 1 & 2 \\ 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{array}
$$

The Markov chain starts at time zero in state $X_0 = 0$. Let

$$T = min\{n \geq 0; X_n = 2\}$$

be the first time that the process reaches state 2. Eventually, the process will reach and be absorbed into state 2. If in some experiment we observed such a process and noted that absorption had not yet taken place, we might be interested in the conditional probability that the process is in state 0 (or 1), given that absorption had not yet taken place. Determine $Pr\{X_3 = 0|X_0, T > 3\}$.

Hint: The event $\{T > 3\}$ is exactly the same as the event $\{X_3 \neq 2\} = \{X_3 = 0\} \cup \{X_3 = 1\}$.

Solution: Let us compute the probability we want as follow:

$$
\begin{aligned}
P\{X_3=0|X_0=0,T>3\} &= \frac{P\{X_3=0,X_0=0,T>3\}}{P\{X_0=0,T>3\}} && \text{conditional prob.}\\
&= \frac{P\{X_3=0,X_0=0,(X_3=0 \text{ OR } X_3=1)\}}{P\{X_0=0,X_3=0 \text{ OR } X_3=1\}} && \text{Following the hint}
\end{aligned}
$$

Analyzing the event on the numerator we have:

$$\{X_3=0,X_0=0,(X_3=0 \text{ OR } X_3=1)\} = \{(X_3=0,X_0=0) \text{ OR } (X_3=0,X_0=0,X_3=1)\} = \{(X_3=0,X_0=0)\}$$

since the second event is impossible (X_3 cannot be 0 and 1 at the same time). For the denominator we have:

$$\{X_0=0,(X_3=0 \text{ OR } X_3=1)\} = \{(X_0=0,X_3=0) \text{ OR } (X_0=0,X_3=1)\}$$

Hence, we obtain

$$
\begin{aligned}
P\{X_3=0|X_0=0,T>3\} &= \frac{P\{(X_3=0,X_0=0)\}}{P\{(X_0=0,X_3=0) \text{ OR } (X_0=0,X_3=1)\}} && \text{conditional prob.}\\
&= \frac{P\{(X_3=0,X_0=0)\}}{P\{(X_0=0,X_3=0)\}+P\{(X_0=0,X_3=1)\}} && \text{Disjoint events}\\
&= \frac{p_{00}^{(3)}}{p_{00}^{(3)}+p_{01}^{(3)}} && \text{3-step transition probability}\\
&= \frac{.457}{(.457+.23)} = \mathbf{0.66521}
\end{aligned}
$$

Where the 3-rd transition probability matrix is obtain by raising the transition probability matrix to the third as follow:

$$\mathbf{P^3} = \left(\begin{Vmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{Vmatrix} \right)^3 = \begin{Vmatrix} 0.457 & 0.230 & 0.313 \\ 0.345 & 0.227 & 0.428 \\ 0 & 0 & 1 \end{Vmatrix}$$