M464 - Introduction To Probability II - Homework 14

Enrique Areyan

May 1, 2014

Chapter 6

Problems

6.1 Let $Y_n,\ n = 0, 1, \ldots,$ be a discrete time, finite Markov chain with transition probabilities $\mathbf{P} = ||P_{ij}||$, and let $\{N(t); t \geq 0\}$ be an independent Poisson process of rate λ. Argue that the compound process

$$X(t) = Y_{N(t)}, \qquad t \geq 0,$$

is a Markov chain in continuous time and determine its infinitesimal parameters.

Solution: That the compound process is an stochastic process in continuous time follows from the definition $X(t) = Y_{N(t)}$, where t is a real-valued parameter. Now, for the probabilities, we compute (for small h):

$$\begin{aligned}
P_{i,j}(t) &= Pr\{X(h) = j | X(0) = i\} && \text{by def. of transition prob.}\\
&= Pr\{Y_{N(h)} = j | Y_{N(0)} = i\} && \text{by def. of } X(t)\\
&= Pr\{Y_{N(h)} = j | Y_0 = i\} && \text{Since } N(t) \text{ is a poisson process it follows } N(0) = 0\\
&= \sum_{k=0}^{\infty} Pr\{Y_k = j | Y_0 = i, N(h) = k\} Pr\{N(h) = k\} && \text{law of total prob.}\\
&= \sum_{k=0}^{\infty} Pr\{Y_k = j | Y_0 = i\} Pr\{N(h) = k\} && \text{equivalent event}\\
&= \sum_{k=0}^{\infty} P_{ij}^{(k)} \frac{e^{-\lambda h}(\lambda h)^k}{k!} && \text{Since } N(h) \sim Pois(\lambda h) \text{ and by transition matrix of } Y_n\\
&= e^{-\lambda h} \sum_{k=0}^{\infty} \frac{P_{ij}^{(k)}(\lambda h)^k}{k!} && \text{Rearranging terms}\\
&= e^{-\lambda h}[0 + \lambda h P_{ij}^{(1)} + o(h)] && \text{Taylor expansion of } e^x \text{ and the fact that } P_{ij}^{(0)} = 0\\
&= e^{-\lambda h} \lambda h P_{ij} + o(h) && \text{Distributive and properties of } o(h)\\
&= \lambda h P_{ij} \left[\sum_{k=0}^{\infty} \frac{(-\lambda h)^k}{k!} + o(h) \right] && \text{Taylor expansion of } e^x\\
&= \lambda h P_{ij} \left[1 + (-\lambda h) + o(h) \right] && \text{Taking first few terms and using } o(h)\\
&= \lambda h P_{ij} + o(h) && \text{Rearranging terms}
\end{aligned}$$

Therefore, $\boxed{q_{ij} = \lambda P_{ij}}$

6.3 Let $X_1(t), X_2(t), \ldots, X_N(t)$ be independent two-state Markov chains having the same infinitesimal matrix

$$\mathbf{A} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} -\lambda & \lambda \\ \mu & -\mu \end{array} \right\| \end{array}$$

Determine the infinitesimal matrix for the Markov chain $Z(t) = X_1(t) + \cdots + X_N(t)$.

Solution: First note that $Z(t) \in \{0, 1, 2, \ldots, N-1, N\}$ since each $X_i(t)$, for $1 \leq i \leq N$, can only either increment to 1 or decrement to 0. Hence, the sum is at least 0 (all $X_i(t) = 0$) or N (all $X_i(t) = 1$). The infinitesimal matrix is a N by N matrix with the rates of change from state i to state j. Now, since these are infinitesimal rates, we can only have numbers distinct from zero when incrementing or decrementing by one or staying the same. For instance, for small h the rate going from state 0 to state 1 can be computed as follow:

$$\begin{aligned} Pr\{Z(h) = 1|Z(0) = 0\} &= Pr\{\text{exactly one } X_i \text{ changes from 0 to 1}\} \\ &= \binom{N}{1}[\lambda h + o(h)](1 - \lambda h - o(h))^{n-1} \\ &= [N\lambda h + o(h)]\left(\sum_{k=0}^{n-1}(1 - \lambda h)^k o(h)^{n-1-k}\right) \\ &= N\lambda h + o(h) \text{ (since all other terms above are } o(h)) \end{aligned}$$

In a similar manner one can compute all other infinitesimal parameters. The values depend mainly on how many $X's$ are in state i at a given time. Hence, the complete infinitesimal matrix is:

$$\mathbf{A}' = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} \left\| \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & N-1 & N \\ -N\lambda & N\lambda & 0 & 0 & \cdots & 0 & 0 \\ \mu & -(\mu + (N-1)\lambda) & (N-1)\lambda & 0 & \cdots & 0 & 0 \\ 0 & 2\mu & -(2\mu + (N-2)\lambda) & (N-2)\lambda & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -((N-1)\mu + \lambda) & \lambda \\ 0 & 0 & 0 & 0 & \cdots & N\mu & -N\mu \end{array} \right\|$$