M464 - Introduction To Probability II - Homework 4

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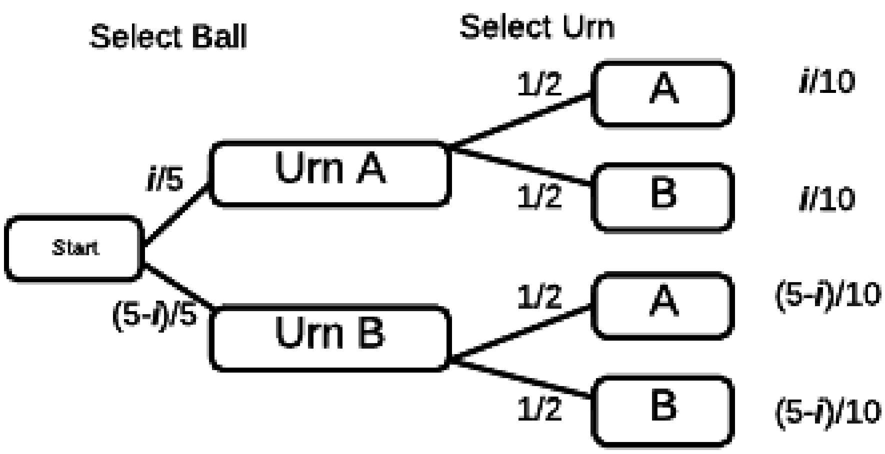
Chapter 4

(1.2) Five balls are distributed between two urns, labeled A and B. Each period, one of the five balls is selected at random, and whichever urn it's in, it is moved to the other urn. In the long run, what fraction of time is urn A empty?
Note: Consider the following modification of the problem so that the resulting chain is regular: when a ball is selected at random, replace it in the urn it came from with probability $1/2$, and otherwise put it in the other urn

Solution: Consider the following model for the experiment described above: Let $X_n = \#$ of balls in urn A at time n. Then $< X_n >$ is a Markov chain with 6 states. The transition probabilities can be computed according to the following tree, where i is the current state of the Markov chain:



,i.e., the probability of staying in the same state is the probability of picking a ball from urn A and leave it there plus the probability of picking a ball from urn B and leave it there: $P_{i,i} = \frac{i}{10} + \frac{5-i}{10} = \frac{5}{10}$. The probability of going from state i to state $i-1$ is the probability of picking a ball from urn A and move it to urn B: $P_{i,i-1} = \frac{i}{10}$. Finally, the probability of moving from state i to state $i+1$ is the probability of picking a ball from urn B and move it to urn A: $P_{i,i+1} = \frac{5-i}{10}$.

Now we can provide the transition probability matrix:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 5/10 & 5/10 & 0 & 0 & 0 & 0 \\ 1/10 & 5/10 & 4/10 & 0 & 0 & 0 \\ 0 & 2/10 & 5/10 & 3/10 & 0 & 0 \\ 0 & 0 & 3/10 & 5/10 & 2/10 & 0 \\ 0 & 0 & 0 & 4/10 & 5/10 & 1/10 \\ 0 & 0 & 0 & 0 & 5/10 & 5/10 \end{array}$$

Note that this is a regular transition probability matrix. We can use the condition given in class to prove this: first, note that there exists a state such that $P_{i,i} > 0$ (in fact all states satisfy this condition so pick any, say $P_{0,0} = 5/10 > 0$). Second, note that we can reach any state j from any other state i via the path $i, i+1, \ldots, j$ or with the reverse path $j, j-1, \ldots, i$. These paths have positive probabilities. Hence, this is a regular transition probability matrix.

Finally, we can find the limiting distribution using Theorem 1.1. and solving the linear system:

$$\pi P = \pi, \text{ and } \sum_{i=0}^{5} \pi_i = 1, \text{ from which we get the equations:}$$

$$\begin{array}{l} \frac{5}{10}\pi_0 + \frac{1}{10}\pi_1 = \pi_0 \\ \frac{5}{10}\pi_0 + \frac{5}{10}\pi_1 + \frac{2}{10}\pi_2 = \pi_1 \\ \frac{4}{10}\pi_1 + \frac{5}{10}\pi_2 + \frac{3}{10}\pi_3 = \pi_2 \\ \frac{3}{10}\pi_2 + \frac{5}{10}\pi_3 + \frac{4}{10}\pi_4 = \pi_3 \\ \frac{2}{10}\pi_3 + \frac{5}{10}\pi_4 + \frac{5}{10}\pi_5 = \pi_4 \\ \frac{1}{10}\pi_4 + \frac{5}{10}\pi_5 = \pi_5 \end{array} \implies \begin{array}{l} 5\pi_0 + \pi_1 = 10\pi_0 \\ 5\pi_0 + 5\pi_1 + 2\pi_2 = 10\pi_1 \\ 4\pi_1 + 5\pi_2 + 3\pi_3 = 10\pi_2 \\ 3\pi_2 + 5\pi_3 + 4\pi_4 = 10\pi_3 \\ 2\pi_3 + 5\pi_4 + 5\pi_5 = 10\pi_4 \\ \pi_4 + 5\pi_5 = 10\pi_5 \end{array} \implies \begin{array}{l} \pi_1 = 5\pi_0 \\ 5\pi_0 + 2\pi_2 = 5\pi_1 \\ 4\pi_1 + 3\pi_3 = 5\pi_2 \\ 3\pi_2 + 4\pi_4 = 5\pi_3 \\ 2\pi_3 + 5\pi_5 = 5\pi_4 \\ \boxed{\pi_4 = 5\pi_5} \end{array}$$

Back substituting the last equation into the others we get: for the fifth eq. $2\pi_3 = 20\pi_5 \Longrightarrow \boxed{\pi_3 = 10\pi_5}$, for the fourth eq. $3\pi_2 + 20\pi_5 = 50\pi_5 \Longrightarrow \boxed{\pi_2 = 10\pi_5}$, for the third eq. $4\pi_1 + 30\pi_5 = 50\pi_5 \Longrightarrow \boxed{\pi_1 = 5\pi_5}$, for the second eq. $5\pi_0 + 20\pi_5 = 25\pi_5 \Longrightarrow \boxed{\pi_0 = \pi_5}$, for the first eq. $\boxed{\pi_1 = 5\pi_5}$. Now we use the equation $\sum_{i=0}^{5} \pi_i = 1$ in terms of π_5:

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1 \Longrightarrow \pi_5 + 5\pi_5 + 10\pi_5 + 10\pi_5 + 5\pi_5 + \pi_5 = 1 \Longrightarrow 32\pi_5 = 1 \Longrightarrow \pi_5 = \frac{1}{32}$$

Hence, in the long run, the fraction of time is urn A empty is $\boxed{\pi_0 = \pi_5 = \frac{1}{32}}$

(1.6) Determine the following limits in terms of the transition probability matri $\mathbf{P} = ||P_{i,j}||$ and limiting distribution $\pi = ||\pi_j||$ of a finite state regular Markov chain $\{X_n\}$:

(a) $\lim_{n\to\infty} Pr\{X_{n+1} = j|X_0 = i\}$

(b) $\lim_{n\to\infty} Pr\{X_n = k, X_{n+1} = j|X_0 = i\}$

(c) $\lim_{n\to\infty} Pr\{X_{n-1} = k, X_n = j|X_0 = i\}$

Solution:

(a) Since the chain is finite and regular this limit is, by definition, the limiting distribution,i.e.,

$$\lim_{n\to\infty} Pr\{X_{n+1} = j|X_0 = i\} = \pi_j$$

Note that $n + 1 \to \infty$ as $n \to \infty$.

(b)

$$\begin{aligned}
\lim_{n\to\infty} Pr\{X_n = k, X_{n+1} = j|X_0 = i\} &= \lim_{n\to\infty} \frac{Pr\{X_n = k, X_{n+1} = j, X_0 = i\}}{Pr\{X_0 = i\}} && \text{conditional probabilities} \\
&= \lim_{n\to\infty} \frac{Pr\{X_{n+1} = j, X_n = k, X_0 = i\}}{Pr\{X_0 = i\}} && \text{Rearranging events} \\
&= \lim_{n\to\infty} \frac{Pr\{X_{n+1} = j|X_n = k, X_0 = i\}Pr\{X_n = k, X_0 = i\}}{Pr\{X_0 = i\}} && \text{conditional probabilities} \\
&= \lim_{n\to\infty} \frac{Pr\{X_{n+1} = j|X_n = k\}Pr\{X_n = k, X_0 = i\}}{Pr\{X_0 = i\}} && \text{Markov property} \\
&= \lim_{n\to\infty} \frac{Pr\{X_{n+1} = j|X_n = k\}Pr\{X_n = k|X_0 = i\}Pr\{X_0 = i\}}{Pr\{X_0 = i\}} && \text{conditional prob.} \\
&= \lim_{n\to\infty} Pr\{X_{n+1} = j|X_n = k\}Pr\{X_n = k|X_0 = i\} && \text{simplifying} \\
&= \lim_{n\to\infty} P_{k,j} Pr\{X_n = k|X_0 = i\} && \text{definition of transition matrix} \\
&= P_{k,j} \lim_{n\to\infty} Pr\{X_n = k|X_0 = i\} && \text{since } P_{k,j} \text{ is a constant} \\
&= P_{k,j} \cdot \pi_k && \text{limiting distribution}
\end{aligned}$$

Hence,

$$\boxed{\lim_{n\to\infty} Pr\{X_n = k, X_{n+1}|X_0 = i\} = P_{k,j} \cdot \pi_k}$$

(c) We can either do the computation as in (b), or note that this is really the same situation as in (b) with the change of variables $m = n - 1$, so that $\lim_{n\to\infty} m = \infty$. Hence,

$$\lim_{n\to\infty} Pr\{X_{n-1} = k, X_n = j|X_0 = i\} = \lim_{m\to\infty} Pr\{X_m = k, X_{m+1} = j|X_0 = i\} = \boxed{P_{k,j} \cdot \pi_k}, \text{ by part (b)}$$

(1.10) *Assume the M.C. is regular.* Consider a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & N \end{array} \\ \left\| \begin{array}{ccccc} p_0 & p_1 & p_2 & \cdots & p_N \\ p_N & p_0 & p_1 & \cdots & p_{N-1} \\ p_{N-1} & p_N & p_0 & \cdots & p_{N-2} \\ \vdots & \vdots & \vdots & & \vdots \\ p_1 & p_2 & p_3 & \cdots & p_0 \end{array} \right\| \end{array}$$

where $0 < p_0 < 1$ and $p_0 + p_1 + \cdots + p_N = 1$. Determine the limiting distribution.

Solution: This transition probability matrix is doubly stochastic Matrix, i.e., $P_{i,j} \geq 0$ and all rows sum to one as well as the rows (by hypothesis). This is because, for an arbitrary column i the first probability is p_i and the final is p_{i+1} (expect for N which start with p_N and ends with p_0), and contains all numbers $p_0, p_1, \ldots p_N$ (though in different order), which we know add to one. The columns have a similar structure starting and ending in different probabilities but nonetheless each column contains the number $p_0, p_1, \ldots p_N$, which, again, we know add to one. Therefore, (assuming the chain is regular) by results derived in class and in section 1.1, the limiting distribution is the uniform distribution on the states of the markov chain, i.e.,

$$\pi = \left(\frac{1}{N+1}, \frac{1}{N+1}, \ldots, \frac{1}{N+1} \right)$$

Indeed we can easily see that $\pi \mathbf{P} = \pi$.

(1.13) A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.4 & 0.4 & 0.2 \\ 0.6 & 0.2 & 0.2 \\ 0.4 & 0.2 & 0.4 \end{array} \right\| \end{array}$$

After a long period of time, you observe the chain and see that it is in state 1. What is the conditional probability that the previous state was state 2?. That is, find

$$\lim_{n \to \infty} Pr\{X_{n-1} = 2 | X_n = 1\}$$

Solution: First, note that this is a regular transition probability matrix. We can reach any state from any other state directly, and there exists a state that can be reach from itself (in fact any state satisfies this condition). Hence, let us find the limiting distribution using Theorem 1.1. and solving the linear system:

$$\pi P = \pi, \text{ and } \sum_{i=0}^{2} \pi_i = 1, \text{ from which we get the equations:}$$

$$\begin{array}{l} \frac{4}{10}\pi_0 + \frac{6}{10}\pi_1 + \frac{4}{10}\pi_2 = \pi_0 \\ \frac{4}{10}\pi_0 + \frac{2}{10}\pi_1 + \frac{2}{10}\pi_2 = \pi_1 \\ \frac{2}{10}\pi_0 + \frac{2}{10}\pi_1 + \frac{4}{10}\pi_2 = \pi_2 \end{array} \implies \begin{array}{l} 4\pi_0 + 6\pi_1 + 4\pi_2 = 10\pi_0 \\ 4\pi_0 + 2\pi_1 + 2\pi_2 = 10\pi_1 \\ 2\pi_0 + 2\pi_1 + 4\pi_2 = 10\pi_2 \end{array} \implies \begin{array}{l} 6\pi_1 + 4\pi_2 = 6\pi_0 \\ 4\pi_0 + 2\pi_2 = 8\pi_1 \\ 2\pi_0 + 2\pi_1 = 6\pi_2 \end{array} \implies \begin{array}{l} 3\pi_1 + 2\pi_2 = 3\pi_0 \\ 2\pi_0 + \pi_2 = 4\pi_1 \\ \pi_0 + \pi_1 = 3\pi_2 \implies \pi_0 = 3\pi_2 - \pi_1 \end{array}$$

Back substituting the last equation into the others we get: $6\pi_2 - 2\pi_1 + \pi_2 = 4\pi_1 \implies 7\pi_2 = 6\pi_1 \implies \pi_2 = \frac{6}{7}\pi_1$. $\pi_0 = 3\left(\frac{6}{7}\pi_1\right) - \pi_1 = \frac{18}{7}\pi_1 - \pi_1 \implies \pi_0 = \frac{11}{7}\pi_1$. Now, using the condition $\pi_0 + \pi_1 + \pi_2 = 1$, we get:

$$\frac{11}{7}\pi_1 + \pi_1 + \frac{6}{7}\pi_1 = 1 \implies \pi_1 \left(\frac{11}{7} + 1 + \frac{6}{7} \right) = 1 \implies \boxed{\pi_1 = \frac{7}{24}}$$

Finally, solve for π_2 and π_0: $\pi_2 = \frac{6}{7}\pi_1 \implies \pi_2 = \frac{6}{7} \cdot \frac{7}{24} \implies \boxed{\pi_2 = \frac{6}{24}}$; $\pi_0 = \frac{11}{7}\pi_1 \implies \frac{11}{7} \cdot \frac{7}{24} \implies \boxed{\pi_0 = \frac{11}{24}}$. The long-term distribution is:

$$\pi = \left(\frac{11}{24}, \frac{7}{24}, \frac{6}{24} \right)$$

Now we can find the following probability:

$$
\begin{aligned}
\lim_{n\to\infty} Pr\{X_{n-1}=2|X_n=1\} &= \lim_{n\to\infty}\frac{Pr\{X_{n-1}=2, X_n=1\}}{Pr\{X_n=1\}} && \text{conditional probabilities}\\
&= \lim_{n\to\infty}\frac{Pr\{X_n=1, X_{n-1}=2\}}{Pr\{X_n=1\}} && \text{Rearranging events}\\
&= \lim_{n\to\infty}\frac{Pr\{X_n=1|X_{n-1}=2\}Pr\{X_{n-1}=2\}}{Pr\{X_n=1\}} && \text{conditional probabilities}\\
&= \lim_{n\to\infty}\frac{P_{2,1}Pr\{X_{n-1}=2\}}{Pr\{X_n=1\}} && \text{definition of transition matrix}\\
&= P_{2,1}\lim_{n\to\infty}\frac{Pr\{X_{n-1}=2\}}{Pr\{X_n=1\}} && \text{since } P_{2,1} \text{ is a constant}\\
&= P_{2,1}\frac{\pi_2}{\pi_1} && \text{lim. distributions (does not depend on initial state)}\\
&= \frac{2}{10}\cdot\frac{\frac{6}{24}}{\frac{7}{24}} = \frac{2}{10}\frac{6}{7} = \frac{12}{70} = \boxed{\frac{6}{35}} && \text{replacing values and multiplying}
\end{aligned}
$$