M464 - Introduction To Probability II - Homework 9

Enrique Areyan
March 27, 2014

Chapter 5

Exercises

3.7 Customers arrive at a service facility according to a Poisson process of rate λ customers/hour. Let $X(t)$ be the number of customers that have arrived up to time t. Let $W_1, W_2, \ldots$ be the successive arrival times of the customers. Determine the conditional mean $E[W_5|X(t) = 3]$.

Solution: We know that up to time t we had 3 customers. Let us compute the density function of $W_5|X(t) = 3$ by first computing its cumulative distribution. Let $u > t$. Then, $Pr\{W_5 \leq u|X(t) = 3\}$ = probability that we will get 5 or more customers past time t and up to time u, given that we had 3 customers up to time t. But this is the same as getting 2 or more customers between time t and u, i.e., $Pr\{W_5 \leq u|X(t) = 3\} = Pr\{X(u) \geq 2|X(t) = 3\}$. Then,

$$\begin{aligned}
Pr\{W_5 \leq u|X(t) = 3\} &= Pr\{X(u) \geq 2|X(t) = 3\} && \text{as explained before} \\
&= Pr\{X(u) - X(t) \geq 2\} && \text{independent increments} \\
&= 1 - Pr\{X(u) - X(t) < 2\} && \text{taking complement} \\
&= 1 - Pr\{X(u) - X(t) = 0 \text{ OR } X(u) - X(t) = 1\} && \text{all possibilities} \\
&= 1 - Pr\{X(u) - X(t) = 0\} - Pr\{X(u) - X(t) = 1\} && \text{disjoint events}
\end{aligned}$$

Since $X(t)$ is a Poisson process, we have that $X(u) - X(t) \sim Pois((u - t)\lambda)$. So we can compute the above probabilities:

$$\begin{aligned}
Pr\{W_5 \leq u|X(t) = 3\} &= 1 - Pr\{X(u) - X(t) = 0\} - Pr\{X(u) - X(t) = 1\} && \text{previous calculation} \\
&= 1 - \frac{e^{-(u-t)\lambda}\left((u-t)\lambda\right)^0}{0!} - \frac{e^{-(u-t)\lambda}\left((u-t)\lambda\right)^1}{1!} && \text{Poisson p.d.f} \\
&= 1 - e^{-(u-t)\lambda} - e^{-(u-t)\lambda}(u - t)\lambda && \text{simplifying}
\end{aligned}$$

Hence, the cdf of $W_5|X(t) = 3$ is $F_{W_5}(u) = 1 - e^{-(u-t)\lambda} - e^{-(u-t)\lambda}(u - t)\lambda$, which means that the pdf is the derivative of this function w.r.t u:

$$f_{W_5}(u) = \frac{d}{du}F_{W_5} = \lambda e^{-(u-t)\lambda} - \lambda e^{-(u-t)\lambda} + (u - t)\lambda^2 e^{-(u-t)\lambda} = (u - t)\lambda^2 e^{-(u-t)\lambda}$$

Finally, we can compute the expectation of this random variable. By definition:

$$\begin{aligned}
E[W_5|X(t) = 3] &= \int_t^\infty u f_{W_5}(u)du && \text{definition of expectation} \\
&= \int_t^\infty u(u - t)\lambda^2 e^{-(u-t)\lambda}du && \text{replacing pdf previously calculated} \\
&= \int_0^\infty (v + t)v\lambda^2 e^{-v\lambda}dv && \text{change of variables } v = u - t \\
&= \lambda^2\left[\int_0^\infty v^2 e^{-v\lambda}dv + t\int_0^\infty ve^{-v\lambda}dv\right] && \text{linearity of integral} \\
&= \lambda^2\left[\left(\frac{e^{-v\lambda}(-\lambda v(\lambda v + 2) - 2)}{\lambda^3}\right) + t\left(\frac{e^{-v\lambda}(\lambda v + 1)}{\lambda^2}\right)\right]_0^\infty && \text{antiderivative}
\end{aligned}$$

Since $\lim_{v\to\infty} e^{-v\lambda} = 0$ and $e^{-\lambda 0} = 1$, we have:

$$E[W_5|X(t)=3] = -\lambda^2\left[\frac{-2}{\lambda^3} - \frac{t}{\lambda^2}\right] = -\left[-\frac{2}{\lambda} - t\right] = \boxed{t + \frac{2}{\lambda}}$$

Note that this result makes intuitive sense: if $\lambda \to 0$, then we would have to wait an arbitrary amount of time past t for customer 5 to occur. Likewise, if $\lambda \to \infty$ then we would have to wait an infinitesimal amount of time past t. Lastly, if $\lambda = 2$, then we would expect to wait exactly one unit of time past t for 2 more customers to arrive and thus, receive the 5th customer.

4.1 Let $\{X(t); t \geq 0\}$ be a Poisson process of rate λ. Suppose it is known that $X(1) = n$. For $n = 1, 2, \ldots,$ determine the mean of the first arrival time W_1.

Solution: Let $U_1, U_2, \ldots, U_n \sim Uniform((0,1])$. Given that $X(1) = n$, the $U's$ represent the $W's$ but ignoring order. Now, we want to find the mean time of W_1, i.e., the first arrival time. In terms of the $U's$ we have that $W_1 = U_{(1)}$, where $U_{(1)} = min\{U_1, U_2, \ldots, U_n\}$. But finding the distribution of $U_{(1)}$ is relatively easy: let $v \in (0,1]$

$$\begin{aligned}
Pr\{W_1 \leq v\} = Pr\{U_{(1)} \leq v\} &= Pr\{min\{U_1, U_2, \ldots, U_n\} \leq v\} && \text{by definition of } U_{(1)} \\
&= 1 - Pr\{min\{U_1, U_2, \ldots, U_n\} > v\} && \text{complementary probability} \\
&= 1 - [Pr\{U_1 > v, U_2 > v, \ldots, U_n > v\}] && \text{minimality of the } U's \\
&= 1 - [Pr\{U_1 > v\}Pr\{U_2 > v\}\cdots Pr\{U_n > v\}] && \text{by independence of the } U's \\
&= 1 - [(1 - Pr\{U_1 \leq v\})(1 - Pr\{U_2 > v\})\cdots(1 - Pr\{U_n > v\})] && \text{complementary probability} \\
&= 1 - [(1-v)(1-v)\cdots(1-v)] && \text{uniform c.d.f} \\
&= 1 - [(1-v)^n]
\end{aligned}$$

Hence, the c.f.d of W_1 is $F_{W_1}(v) = 1 - [(1-v)^n]$, thus the p.d.f if $\frac{d}{dv}F_{W_1}(v) = n(1-v)^{n-1} = f_{W_1}(v)$. The mean is:

$$\int_0^1 v f_{W_1}(v)dv = \int_0^1 v \cdot n(1-v)^{n-1}dv$$

Make the change of variables $1 - v = z \Longrightarrow v = 1 - z$. Thus, if $v = 0$ then $z = 1$ and if $v = 1$ then $z = 0$.

$$\int_0^1 v f_{W_1}(v)dv = -n\int_1^0 (1-z)\cdot z^{n-1}dz = -n\int_1^0 z^{n-1} - z^n dz = -n\left[\frac{z^n}{n} - \frac{z^{n+1}}{n+1}\right]_1^0 = -n\left[-\frac{1}{n} + \frac{1}{n+1}\right] = 1 - \frac{n}{n+1} = \boxed{\frac{1}{n+1}}$$

4.2 Let $\{X(t); t \geq 0\}$ be a Poisson process of rate λ. Suppose it is known that $X(1) = 2$. Determine the mean of W_1W_2, the product of the first two arrival times.

Solution: Let $U_1, U_2 \sim Uniform((0,1])$. Since we know that $X(1) = 2$, the $U's$ represent the $W's$ but ignoring order. Since the product of two real numbers is conmutative, we have that $U_1U_2 = W_1W_2$, i.e., it does not matter if we multiply the ordered or unordered random variables, the result will be the same. But computing expected values of the $U's$ is very easy: $E[U_1] = E[U_2] = \frac{1}{2}$ (mean of a uniform on $(0,1]$, i.e., $\int_0^1 u du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$). Finally, since U_1 is independent of U_2:

$$E[W_1W_2] = E[U_1U_2] = E[U_1]E[U_2] = \frac{1}{2}\cdot\frac{1}{2} = \boxed{\frac{1}{4}}$$

4.5 Customers arrive at a certain facility according to a Poisson process of rate λ. Suppose that it is known that five customers arrived in the first hour. Each customer spends a time in the store that is a random variable, exponentially distributed with parameter α and independent of the other customer times, and then departs. What is the probability that the store is empty at the end of this first hour?

Solution: Mathematically, this problem models the same situation as that of decaying particles worked in class. In this case let us interpret a particle being alive at time t as a customer being in the store at time t. Hence, the probability of a customer being at the store at time t is given by:

$$p = 1 - \frac{1}{t}\int_0^t G(v)dv$$

In this case the distribution of the time spent in the store by each customer is a exponentially distributed, i.e., $G(v) = 1 - e^{-\alpha v}$. Solving for p up to time $t = 1$:

$$p = 1 - \frac{1}{t}\int_0^t G(v)dv = 1 - \frac{1}{1}\int_0^1 1 - e^{-\alpha v}dv = 1 - 1 - \left[\frac{e^{-\alpha v}}{\alpha}\right]_0^1 = -\left(\frac{e^{-\alpha}}{\alpha} - \frac{e^0}{\alpha}\right) = \frac{1 - e^{-\alpha}}{\alpha}$$

The complement of this probability is the probability that the customer is *not* at the store at time t:

$$1 - p = 1 - \frac{1 - e^{-\alpha}}{\alpha}$$

Since each customer spends a time in the store that is independent of the other customers, the probability that the store is empty at the end of the first hour is the product of these probabilities, i.e.: $Pr\{\text{store empty at tend of first hour}\} = Pr\{\text{customer 1 leaves before first hour, customer 2 leaves before first hour}, \ldots, \text{customer 5 leaves before first hour}\} = Pr\{\text{customer 1 leaves before first hour}\}Pr\{\text{customer 2 leaves before first hour}\}\cdots Pr\{\text{customer 5 leaves before first hour}\} = 1 - \frac{1 - e^{-\alpha}}{\alpha} \cdot 1 - \frac{1 - e^{-\alpha}}{\alpha} \cdots 1 - \frac{1 - e^{-\alpha}}{\alpha} = \boxed{\left[1 - \frac{1 - e^{-\alpha}}{\alpha}\right]^5}$

Problems

3.7 A critical component on a submarine has an operating lifetime that is exponentially distributed with mean 0.50 years. As soon as a component fails, it is replaced by a new one having statistically identical properties. What is the smallest number of *spare* components that the submarine should stock if it leaving for a one-year tour and wishes the probability of having an inoperable unit caused by failures exceeding the spare inventory to be less than 0.02?

Solution: Let $X(t)$ = number of component failures (equivalently, this is the same as counting the number of component replacements since we replace components as soon as it fails). Now, by theorem 3.2, we know that $X(t)$ is a Poisson process in this case of rate $\lambda = \frac{\text{1 failure}}{\text{1/2 year}}$. Hence, $X(t) \sim Pois(\frac{\text{1 failure}}{\text{1/2 year}} \cdot t)$.

We are interested in one year so we will use $X(1) \sim Pois(\frac{\text{1 failure}}{\text{1/2 year}} \cdot 1) = Pois(2)$. We wish to minimizing the probability:

$$Pr\{X(1) = n\} \leq 0.02$$

Calculating this probability for various values of n we find that:

$$n = 6 \Longrightarrow \quad Pr\{x(1) = 6\} = \frac{e^{-2}2^6}{6!} = \frac{0.13533528323 \cdot 64}{720} = 0.01202980295$$

$$n = 7 \Longrightarrow \quad Pr\{x(1) = 7\} = \frac{e^{-2}2^7}{7!} = \frac{0.13533528323 \cdot 128}{5040} = 0.00343708655$$

Clearly values from $n = 1$ up to $n = 5$ have higher probability than 0.01202980295, while values bigger than $n = 7$ will have lower probability than 0.00343708655. Therefore, the value of n that minimizes the above probability is $n = 7$. This means that there must be $\boxed{\text{6 } \textit{spare} \text{ units}}$ for the probability of having an inoperable unit caused by failures exceeding the spare inventory to be less than 0.02. Note that one component is loaded at the beginning of the voyage and does not count as a *spare* unit.

4.8 Electrical pulses with independent and identically distributed random amplitude $\xi_1, \xi_2, \ldots$ arrive at a detector at random times $W_1, W_2, \ldots$ according to a Poisson process of rate λ. The detector output $\theta_k(t)$ for the kth pulse at time t is

$$\theta_k(t) = \begin{cases} 0 & \text{for } t < W_k \\ \xi_k e^{-\alpha(t - W_k)} & \text{for } t \geq W_k \end{cases}$$

That is, the amplitude impressed on the detector when the pulse arrives is ξ_k, and its effect thereafter decays exponentially at rate α. Assume that the detector is additive, so that if $N(t)$ pulses arrive during the time interval $[0, t]$, then the output at time t is

$$Z(t) = \sum_{k=1}^{N(t)} \theta_k(t)$$

Determine the mean output $E[Z(t)]$ assuming $N(0) = 0$. Assume that the amplitudes $\xi_1, \xi_2, \ldots$ are independent of the arrival times $W_1, W_2, \ldots$

Solution:

$$\begin{aligned} E[Z(t)] &= E\left[\sum_{k=1}^{N(t)} \theta_k(t)\right] && \text{by definition of } Z(t) \\ &= \sum_{n=1}^{\infty} E\left[\sum_{k=1}^{n} \theta_k(t)|N(t)=n\right] Pr\{N(t)=n\} && \text{law of total expectation} \end{aligned}$$

Let us compute, for a fixed n the following expectation. Note that $U_1, \ldots, U_n$ denote independent random variables that are uniformly distributed in $[0, t]$:

$$\begin{aligned} E\left[\sum_{k=1}^{n} \theta_k(t)|N(t)=n\right] &= E\left[\sum_{k=1}^{n} \xi_k e^{-\alpha(t-W_k)}|N(t)=n\right] && \text{Definition of } \theta_k(t) \\ &= E\left[\sum_{k=1}^{n} \xi_k e^{-\alpha(t-U_k)}\right] && \text{Order does not matter in sum and Theorem 4.1} \\ &= nE[\xi_k]E\left[e^{-\alpha(t-U_k)}\right] && \text{Linearity of expectation and independence of } \xi's \text{ with } W's \\ &= nE[\xi_k]\frac{1}{t}\int_0^t e^{-\alpha(t-u)}du && \text{Law of unconscious statistician} \\ &= nE[\xi_k]\frac{e^{-\alpha t}}{t}\int_0^t e^{\alpha u}du && \text{Taking constants out of integral} \\ &= nE[\xi_k]\frac{e^{-\alpha t}}{t}\left[\frac{e^{\alpha u}}{\alpha}\right]_0^1 && \text{Integrating} \\ &= nE[\xi_k]\frac{e^{-\alpha t}}{t}\left[\frac{e^{\alpha t}-1}{\alpha}\right]_0^1 && \text{Evaluating limits} \\ &= \frac{n}{t}E[\xi_k]\left[\frac{1-e^{-\alpha t}}{\alpha}\right] && \text{algebra} \end{aligned}$$

Plugging back into the expectation we want:

$$\begin{aligned} E[Z(t)] &= \sum_{n=1}^{\infty} \frac{n}{t}E[\xi_k]\left[\frac{1-e^{-\alpha t}}{\alpha}\right] Pr\{N(t)=n\} && \text{replacing} \\ &= \frac{E[\xi_k]}{t}\left[\frac{1-e^{-\alpha t}}{\alpha}\right]\sum_{n=1}^{\infty} nPr\{N(t)=n\} && \text{taking constants out of sum} \\ &= \frac{E[\xi_k]}{t}\left[\frac{1-e^{-\alpha t}}{\alpha}\right]E[N(t)] && \text{By definition of expectation of a discrete r.v.} \\ &= \frac{E[\xi_k]}{t}\left[\frac{1-e^{-\alpha t}}{\alpha}\right]\lambda t && \text{Since } N(t) \sim Pois(\lambda t) \\ &= \frac{\lambda E[\xi_k](1-e^{-\alpha t})}{\alpha} && \text{Simplifying and rearranging terms} \end{aligned}$$

Thus, we found the quantity: $\boxed{E[Z(t)] = \frac{\lambda E[\xi_k](1-e^{-\alpha t})}{\alpha}}$