CHAPTER V: Poisson Processes.

1.1 The Poisson Distribution: $X \sim Pois(\mu)$ $(\mu > 0)$ if $P(X=k) = e^{-\mu}\frac{\mu^k}{k!}$ $(k \in \mathbb{N})$.
$E[X] = \mu = Var[X]$; $E[X^2] = \mu + \mu^2$

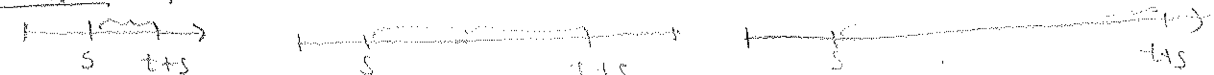
THEOREM 1.1: Let $X \sim Pois(\mu)$, $Y \sim Pois(\nu)$ be independent. Then $X+Y \sim Pois(\mu+\nu)$.

THEOREM 1.2: Let $N \sim Pois(\mu)$, $M|N \sim Bin(N,p)$ then, unconditional dist of M is $M \sim Pois(\mu p)$.

1.2 The Poisson Process:

*Independent increments: we say that a stochastic process $\langle X(t); t \geq 0 \rangle$ has independent increments if $\forall\ 0 = t_0 < t_1 < \dots < t_n$: the r.v.'s $X(t_{k+1}) - X(t_k)$ $(0 \leq k < n)$ are independent.

*Stationary increments: we say that a stochastic process $\langle X(t); t \geq 0 \rangle$ has stationary increments if $\forall t > 0$ the distribution of $X(s+t) - X(s)$ does not depend on s.



Definition: A stochastic process $\langle X(t); t \geq 0 \rangle$ is called a Poisson process of intensity (or rate) λ if the following five conditions hold:

Note: $X(t)$ counts how many events up to time t

(i) $\lambda > 0$
(ii) $\forall t$: $X(t) \in \mathbb{N}$
(iii) $\langle X(t) \rangle$ has independent, stationary increments
(iv) $X(0) = 0$
(v) $\forall t > 0$: $X(t) \sim Pois(\lambda t)$

Note: if $0 < s < t$: $X(t) - X(s) \sim Pois(\lambda(t-s))$

If $\langle X(t): t \geq 0 \rangle$ is a Poisson process, then $E[X(t)] = \lambda t = Var[X(t)]$.

Note: $s < t$: $X(s), X(t) \Rightarrow$ are not independent
$X(s), X(t) - X(s) \Rightarrow$ are independent

1.3 Nonhomogeneous Poisson Processes

Definition: A stochastic process $\langle X(t); t \geq 0 \rangle$ is called a non homogeneous Poisson process of intensity (or rate) function $\lambda(t)$ if the following five cond. hold:

Note: what if t is very small: $X(s+ds) - X(s) \sim Pois(\lambda(s)ds)$

(i) $\forall t \geq 0$ $\lambda(t) \geq 0$
(ii) $\forall t$: $X(t) \in \mathbb{N}$
(iii) $\langle X(t) \rangle$ has independent increments
(iv) $X(0) = 0$
(v) $\forall t > 0, \forall s \geq 0$: $X(s+t) - X(s) \sim Pois\left(\int_s^{s+t} \lambda(u)\,du\right)$

Note: A change of scale converts any non-hom. process into a hom pro.

2. The Law of rare events: $Bin(n, \frac{\mu}{n}) \Rightarrow Pois(\mu)$ as $n \to \infty$ [i.e., $\forall k \in \mathbb{N}$: $P[Bin(n, \frac{\mu}{n}) = k] \to P[Pois(\mu) = k]$]

This also works in general for different values of p.

Thm: Suppose that $\forall n$: $X_{n,i}$ $(1 \leq i \leq n)$ are independent r.v. with values in $\mathbb{N}$. Let $p_{n,i} := P[X_{n,i} = 1]$, $\varepsilon_{n,i} := P[X_{n,i} \geq 2]$ satisfy $\sum_{i=1}^{n} p_{n,i} \to \mu \in [0, \infty)$, $\max_{1 \leq i \leq n} p_{n,i} \to 0$ and $\sum_{i=1}^{n} \varepsilon_{n,i} \to 0$. Then $\sum_{i=1}^{n} X_{n,i} \Rightarrow Pois(\mu)$ [Here $Pois(0)$ means 0 and $Pois(\infty)$ means ∞].

2.1 Ubiquity of Poisson Processes: We call $\langle N(t); t \geq 0 \rangle$ a counting process if $N(t)$ is the finite number of "events" that occur in $(0,t]$. Thus, $N(t) - N(s)$ (for $s < t$) $= \#$ events in $(s,t]$. Formally $N(t) \in \mathbb{N}$. If $s < t$ then $N(s) \leq N(t)$, and $N(\cdot)$ is right continuous (with prob. 1).

THM : Suppose that $N(\cdot)$ is a counting process s.t. (i) it has independent, stationary increments, (ii) it never jumps by more than 1, (iii) $N(0) \equiv 0$. Then, $\exists\ \lambda \in [0,\infty)$ s.t. $\forall t : N(t) \sim Pois(\lambda t)$. So $N(\cdot)$ is a Poisson process of rate λ.

3. Waiting times: Let W_n be the time of the n^{th} event. $W_n := \min\{t; X(t)=n\} = \min\{t; X(t) \geq n\}$

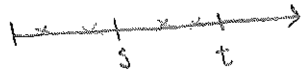
THM : W_n has gamma distribution $\Gamma(n,\lambda)$ with density $(\lambda^n t^{n-1} e^{-\lambda t})/(n-1)!$ $(t \geq 0)$.

In particular : $W_1 \sim Exp(\lambda)$; $f_{W_1}(t) = \lambda e^{-\lambda t}$. Pf : $P[W_n \in (t, t+dt)] = P[X(t) = n-1, X(t+dt) - X(t) = 1]$
$= P[X(t)=n-1]\, P[X(t+dt)-X(t)=1] = (e^{-\lambda t}(\lambda t)^{n-1})/(n-1)! \cdot (e^{\lambda dt} (\lambda dt)^1)/1! \Rightarrow (e^{-\lambda t}(\lambda t)^{n-1})/(n-1)!$

Let $S_n := W_{n+1} - W_n$ be the Sojourn or interarrival times

THM : S_k are IID $Exp(\lambda)$; So $f_{S_k}(t) = \lambda e^{-\lambda t}$; $t \geq 0$.

THM : If $s<t$, then $X(s) \mid X(t) = n \sim Bin(n, \frac{s}{t})$.



4. Relation to Uniform Distribution:

THM: Let $N(\cdot)$ be a Poisson process of rate $\lambda > 0$. Given that $N(t) = n$, the waiting times $W_1, \dots, W_n$ have joint density : $f_{W_1,\dots,W_n}(w_1,\dots,w_n) = \begin{cases} n!\, t^{-n} & \text{if } 0 \leq w_1 \leq \dots \leq w_n \\ 0 & \text{otherwise.} \end{cases}$

[Given # of events up to time t, $W_1, \dots, W_n$ can be represented as $U_1, \dots U_n \sim$ Uniform $[0,t]$]

So, W_i are ordered, U_i are not ordered. If we sum, order does not matter!.

5. Spatial Poisson Processes :

Let $S \subseteq \mathbb{R}^d$. A point process in S is a stochastic process $N(\cdot)$ indexed by subsets $A \subseteq S$, s.t. $N(A) \in \mathbb{N}$ and if $A \cap B = \emptyset \Rightarrow N(A \cup B) = N(A) + N(B)$.

A homogeneous Poisson point process of intensity λ is a point process s.t. $N(A) \sim Pois(\lambda |A|)$ and $A_1, \dots, A_N$ disjoint $\Rightarrow N(A_1), \dots, N(A_N)$ independent. This holds iff: (iii) independence (i) the distribution of $N(A)$ depends only on $|A|$. (ii) $\frac{1}{|A|} P(N(A) \geq 1) \to \lambda$ as $|A| \to 0$ (iv) $P(N(A) \geq 2)/P(N(A)=1) = P(N(A) \geq 2 \mid N(A) = 1) \to 0$ as $|A| \to 0$ (Points don't come in pairs).

THM: If $N(\cdot)$ is a Poisson point process in S, then $\forall A \subseteq S$, the unordered points in A given $N(A) = n$ are i.i.d Uniform on A.

THM : Suppose A is partitioned as $\bigcup_{i=1}^{m} A_i$. Then, The joint distribution of $N(A_1), \dots, N(A_m)$ given $N(A)=n$ is multinomial with parameters $(n; \frac{|A_1|}{|A|}, \dots, \frac{|A_m|}{|A|})$.

6. thinned Poisson Processes: Suppose each event of a $Pois(\lambda)$ process is classified as Type I or Type II independently with prob. p or $1-p$. Then, $X_I(\cdot)$, $X_{II}(\cdot)$ are independent Poisson processes with rates $p\lambda$ and $(1-p)\lambda$.

NoTe : This works the other way as well: If you have two independent Pois. processes of rate a and b, then the combined process is Poisson of rate $a+b$.

CHAPTER VI: Continuous time MCS.

STATE SPACE STILL discrete. transitions occur at real times.
"Timers" with rates $q_{ij} = q_i P_{ij}$; the first Timer to ring among the ones coming from state i is the transition made.

$\min(\underbrace{Exp(\mu), Exp(\lambda)}) = Exp(\lambda+\mu)$
↳ minimum of two with prob $\frac{\mu}{\mu+\lambda}$

At state i, there is a "timer", when it rings", the state changes from i, i.e. the chain stays at i for a random time of length $Exp(q_i)$

the rate of leaving i is $\sum_{j\neq i} q_{ij} = \sum_{j\neq i} q_i P_{ij} = q_i$, and it moves to j with prob $q_{ij}/\sum_{k\neq i} q_{ik} = q_i P_{ij}/q_i = P_{ij}$

Notes: 1) In state diagram: no arrows back to a state from itself.
2) How long you stay in a state depends on i, unlike discrete M.C.
3) the memory less property of Exp distribution makes it a M.C.

Pure Birth Processes: $q_{i,j} = 0$ for $j \neq i+1$ (only increment by one)

$$\lambda_i := q_{i,i+1} = q_i$$

[Poisson process are examples: All $q_i = \lambda$: $P_{i,i+1} = 1$ or $q_{i,i+1} = \lambda$, $q_{i,j} = 0$ $j \neq i+1$]

Yule Process: $\lambda_k = k\beta$ and $X(0) = 1$. All members of the population have the same birth rate β forever, independently of each other.

Pure death Processes: 0 is an absorbing state. Some N is the starting state. this is a pure birth process if we label states in reverse.

Birth and Death Processes: MC with $P_{ij} = 0$ if $|i-j| > 1$

Limit behavior of Birth & Death M.C:

$$\theta_0 = 1, \quad \theta_j = \frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\cdots\mu_j}, \quad j \geq 1 \Rightarrow \quad \pi_j = \frac{\theta_j}{\sum_{k=0}^{\infty}\theta_k}, \quad j = 0,1,2,\ldots$$

Finite State MCs: $\underset{\sim}{A} = \lim_{h\to 0} \frac{\underset{\sim}{P}(h) - I}{h} = \underset{\sim}{P}'(0)$

(i,j)-entry of $\underset{\sim}{A}$ is $\begin{cases} q_{ij} & \text{for } i \neq j \\ -q_i & \text{for } i = j \end{cases}$

EX: Draftsman: $\underset{\sim}{A} = \begin{array}{c|ccc} & T & J & S \\ \hline T & -10 & 10 & 0 \\ J & 2 & -5 & 3 \\ S & 1 & 0 & -1 \end{array}$

Fill in diagonal so that sum of the rows is zero. Solve it as in discrete case but $\pi \underset{\sim}{A} = 0$

CHAPTER 8: Brownian Motion.

A ctn. stochastic process $X(\cdot)$ with stationary independent increments that satisfies: $\exists \mu \in \mathbb{R}$; $\exists \sigma > 0$; $\forall t \geq 0$ $X(t) - X(0) \sim N(\mu t, \sigma^2 t)$ is called Brownian Motion (B.M) with drift μ and variance parameter σ^2 (a.k.a. diffusion coefficient).

If $X(0) = 0$, $\mu = 0$, $\sigma = 1$, then this is std B.M.

NOTE that in general, $B(t) := \frac{(X(t)-X(0)) - \mu t}{\sigma} \sim N(0,t)$ is std B.M

AND if $B(\cdot)$ is std B.M:

then $X(t) := X(0) + \mu t + \sigma B(t)$ is B.M. with drift μ and var par. σ^2.

For all $s,t \geq 0$, $Cov(B(s),B(t)) = \sigma^2 \min\{s,t\}$

$Cov(B(s),B(t)) = E[B(s)B(t)] = E[B(s)((B(t)-B(s)) + B(s))]$
$0 \leq s < t$

$= E[B(s)^2 + [B(s)(B(t)-B(s))]]$

$= E[B(s)^2] + \underbrace{E[B(s)]}_{0} E[B(t)-B(s)]$

$= \boxed{s}$

Note: All odd powers of normal r.v. are zero, all even power have to be computed.