S520 Homework 4

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Chapter 5, Section 6, #1:

(a) i. p is a real number

ii. P is a function with the following signature: $P : R \mapsto R$

iii. Z is a random variable, which means is a function, i.e., $Z : S \mapsto R$. The sample space is usually $S = R$

(b) i. σ is a real number

ii. E is a real number (however, one can think of E also as a function from the set of all random variables to R).

iii. X is a random variable, which means is a function, i.e., $Z : S \mapsto R$.

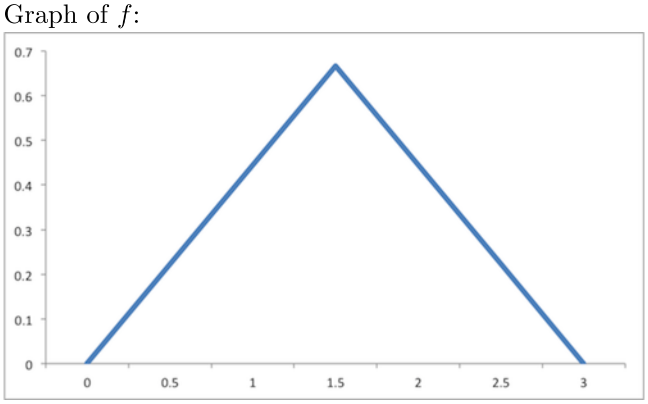
iv. μ is a real number

Chapter 5, Section 6, #3:

(a) For f to be a p.d.f, it has to be the case that $f(x) \geq 0$ for all $x \in R$, and $\int\limits_{-\infty}^{\infty} f(x)dx = 1$. Therefore:

$$\begin{aligned} 1 &= \int\limits_0^{\frac{3}{2}} cxdx + \int\limits_{\frac{3}{2}}^{3} c(3-x)dx \\ &= c\{\left|\frac{x^2}{2}\right|_0^{\frac{3}{2}} + \left|3x - \frac{x^2}{2}\right|_{\frac{3}{2}}^{3}\} \\ &= c\{\frac{9}{8} + [(9 - \frac{9}{2}) - (\frac{9}{2} - \frac{9}{8})]\} \\ &= c\{\frac{9}{8} + [\frac{9}{2} - \frac{25}{8}]\} \\ &= c\{\frac{9}{8} + \frac{9}{8}\} \\ &= c\frac{9}{4} \Longrightarrow c = \frac{4}{9} \end{aligned}$$

(b) Graph of f:



By inspecting the graph of f, we can conclude that $EX = 1.5$
One can also check this result using calculus:

$$
\begin{aligned}
EX &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{-\infty}^{0} xf(x)dx + \int_{0}^{1.5} xf(x)dx + \int_{1.5}^{3} xf(x)dx + \int_{3}^{\infty} xf(x)dx \\
&= 0 + \int_{0}^{1.5} x(\tfrac{4}{9}x)dx + \int_{1.5}^{3} x(\tfrac{4}{9}(3-x))dx + 0 \\
&= \tfrac{4}{9}\{\int_{0}^{1.5} x^2dx + \int_{1.5}^{3} x(3x-x^2)dx\} \\
&= \tfrac{4}{9}\{\left|\tfrac{x^3}{3}\right|_{0}^{1.5} + \left|\tfrac{3x^2}{2}\right|_{1.5}^{3} - \left|\tfrac{x^3}{3}\right|_{1.5}^{3}\} \\
&= \tfrac{4}{9}\{[\tfrac{\frac{3}{2}^3}{3}] + [\tfrac{3^3-3\frac{3}{2}^2}{2}] - [\tfrac{3^3-\frac{3}{2}^3}{3}]\} \\
&= \tfrac{4}{9}\{\tfrac{9}{8} + \tfrac{81}{8} - \tfrac{63}{8}\} \\
&= \tfrac{4}{9}\{\tfrac{27}{8}\} \\
&= \tfrac{3}{2} = 1.5
\end{aligned}
$$

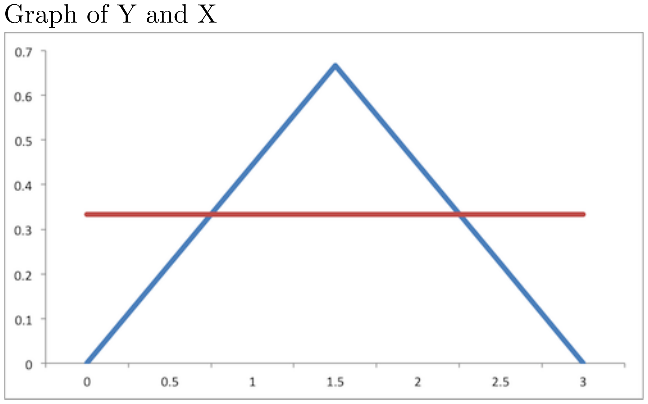
(c)

$$P(X > 2) = 1 - (A_{(0,1.5)} + A_{(1.5,2)})$$

From the graph we know that $A_{(0,1.5)} = \frac{1}{2}$. Now, the area $A_{(1.5,2)}$ can be further decompose as $A_r + A_t$, where A_r is the area of the rectangle with base from 1.5 to 2 and height $f(2)$, i.e. $A_r = 0.5 \cdot \frac{4}{9} = \frac{2}{9}$; and A_t is the area of the triangle with the same base and height $f(1.5) - f(2)$, i.e. $A_t = \frac{0.5 \cdot \frac{2}{3} - \frac{4}{9}}{2} = \frac{1}{18}$. Therefore:

$$P(X > 2) = 1-(\frac{1}{2}+A_{(1.5,2)}) = 1-\frac{1}{2}-A_r-A_t = \frac{1}{2}-\frac{2}{9}-\frac{1}{18} = \frac{4}{18} = \frac{2}{9}$$

(d) Graph of Y and X



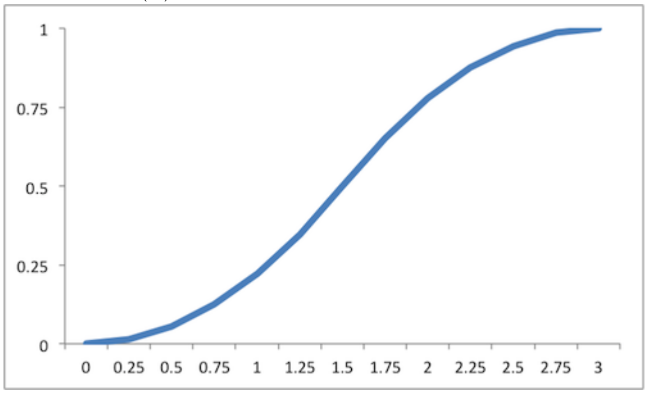
By inspecting the graph above one can conclude that $VarY > VarX$

(e) The values of $F(x)$ were obtained by integrating $f(x)$ appropiately:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{2}{9}x^2 & \text{if } 0 < x < 1.5 \\ (\frac{4}{9}(3x - \frac{x^2}{2})) - 1 & \text{if } 1.5 < x < 3 \\ 1 & \text{if } x > 3 \end{cases}$$

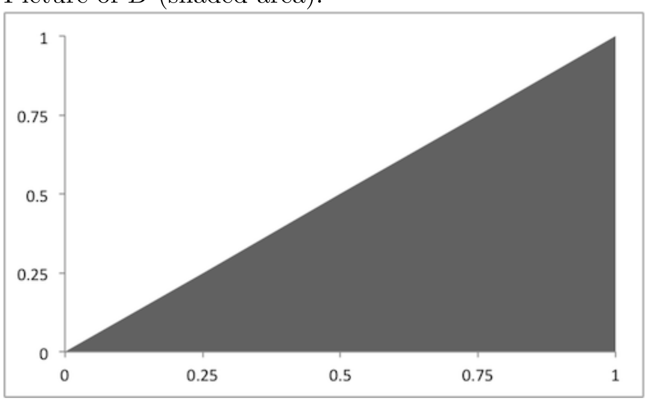
With this function, we can calculate: $P(X > 2) = 1 - F(2) = 1 - \frac{7}{9} = \frac{2}{9}$. Same answer obtained in (c).

Graph of $F(x)$:



Chapter 5, Section 6, #5:

(a) Picture of B (shaded area):



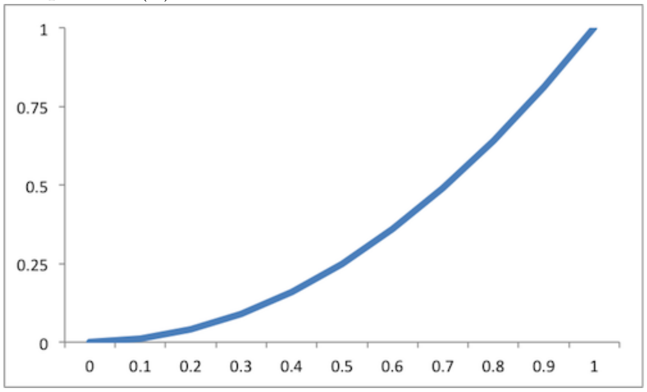
(b) The probability $P(X \leq 0.5)$ is the proportion of points that lie in the triangle from 0 to 0.5 relative to all points, i.e., $P(X \leq 0.5) = \frac{\frac{1}{8}}{\frac{1}{8}+\frac{1}{8}+\frac{1}{4}} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{2}{8} = \frac{1}{4}$. This result is confirmed by the c.d.f calculated for the next question.

(c)

To have a valid c.d.f for$X$, the values must go from 0 to 1. Thus, we need to rescale the above graph. Using calculus, we want the are to sum to one, so we integrate and multiply by a constant 2. F result as follows:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Graph of $F(x)$ for r.v. X



(d) No, X and Y are not independent.
To show that X and Y are not independent, let us work a counter example. By definition, $P(Y = y \in (0.75, 1)|X \in (0, 0.5)) = 0$, but $P(Y = y \in (0, 0.5)) > 0$. Hence, the two r.v. are not independent.

Chapter 5, Section 6, #7:

$$X \sim Normal(\mu = -5, \sigma = 10)$$

(a)

$$\begin{aligned} P(X < 0) &= P(\tfrac{X-\mu}{\sigma} < \tfrac{0-\mu}{\sigma}) \\ &= P(Z < \tfrac{5}{10}) \\ &= P(Z < \tfrac{1}{2}) \\ &= pnorm(0.5) = 0.6914625 \end{aligned}$$

(b)

$$\begin{aligned} P(X > 5) &= P(\tfrac{X-\mu}{\sigma} > \tfrac{5-(-5)}{10}) \\ &= P(Z > 1) \\ &= 1 - P(Z \leq 1) \\ &= 1 - pnorm(1) = 0.1586553 \end{aligned}$$

(c)

$$\begin{aligned} P(-3 < X < 7) &= P(\tfrac{-3+5}{10} < Z < \tfrac{7+5}{10}) \\ &= P(\tfrac{1}{5} < Z < \tfrac{6}{5}) \\ &= P(Z < \tfrac{6}{5}) - P(Z < \tfrac{1}{5}) \\ &= pnorm(1.2) - pnorm(0.2) = 0.3056706 \end{aligned}$$

(d)

$$\begin{aligned} P(|X+5|<10) &= P(-10<X+5<10) \\ &= P(-15<X<5) \\ &= P(X<5)-P(X<-15) \\ &= P(Z<1)-P(Z<-1) \\ &= pnorm(1)-pnorm(-1)=0.6826895 \end{aligned}$$

(e)

$$\begin{aligned} P(|X-3|>2) &= P(X-3>2 \text{ or } X-3<-2) \\ &= P(X-3>2)+P(X-3<-2) \\ &= 1-P(X<5)+P(X<1) \\ &= 1-P(Z<0)+P(Z<-0.4) \\ &= 1-pnorm(0)+pnorm(-0.4)=0.8445783 \end{aligned}$$

Chapter 6, Section 4, #2:

(a) f is a p.d.f iff $f(x)\geq 0, \forall x\in R$, and $\int_{-\infty}^{\infty} f(x)dx=1$. Therefore,

$$\begin{aligned} 1 &= \int_0^1 cxdx+\int_1^2 cdx+\int_2^3 c(3-x)dx \\ &= c\left|\frac{x^2}{2}\right|_0^1+c\left|x\right|_1^2+c\left|3x-\frac{x^2}{2}\right|_2^3 \\ &= \frac{c}{2}+\{2c-c\}+\{9c-\frac{9c}{2}-6c+2c\} \\ &= \frac{c}{2}+c+5c-\frac{9c}{2} \\ &= -4c+6c \Longrightarrow 2c=1 \Longrightarrow c=\frac{1}{2} \end{aligned}$$

(b) $P(1.5<X<2.5)$. To answer this question I first compute the c.d.f of f. Using calculus:

$$F(x)=\begin{cases} 0 & \text{if } x<0 \\ \frac{x^2}{4} & \text{if } x\in[0,1] \\ \frac{x}{2}-\frac{1}{4} & \text{if } x\in[1,2] \\ \frac{1}{2}(3x-\frac{x^2}{2})-\frac{5}{4} & \text{if } x\in[2,3] \\ 1 & \text{if } x>3 \end{cases}$$

$$\begin{aligned} P(1.5<X<2.5) &= P(X<2.5)-P(X<1.5) \\ &= F(2.5)-F(1.5) \\ &= \frac{1}{2}(3\frac{10}{4}-\frac{(\frac{10}{4})^2}{2})-\frac{5}{4}-(\frac{1}{2}\frac{3}{2}-\frac{1}{4}) \\ &= 0.4375 \end{aligned}$$

(c) By inspecting the graph of f, one can easily see that $EX=\frac{3}{2}$. This result is confirmed by calculus:

$$\begin{aligned} EX &= \int_0^1 \frac{x^2}{2}dx+\int_1^2 \frac{x}{2}dx+\int_2^3 \frac{1}{2}(3x-x^2)dx \\ &= \left|\frac{x^3}{6}\right|_0^1+\left|\frac{x^2}{4}\right|_1^2+\left|\frac{1}{2}(\frac{3x^2}{2}-\frac{x^3}{3})\right|_2^3 \\ &= \frac{1}{6}+\frac{3}{4}+\{\frac{1}{2}[(\frac{3^3}{2}-\frac{3^3}{3})-(\frac{3(2^2)}{2}-\frac{2^3}{3})]\} \\ &= \frac{11}{12}+\{\frac{1}{2}[\frac{3^4-3^22-3^22^2+2^4}{6}]\} \\ &= \frac{11+81-54-36+16}{12}=\frac{92-90+16}{12}=\frac{18}{12}=\frac{3}{2} \end{aligned}$$

(d) Using the c.d.f of f previously calculated, the answer is $F(1) = \frac{1}{4}$

(e) We want to find y such that $F(y) = 0.9$. Solving this equation is equivalent to solving the following quadratic equation:

$$-\frac{x^2}{2} + 3x - 4.3 = 0$$

Out of the two roots, the appropiate answer is approximately 2.367544467

Chapter 6, Section 4, #6:

(a) $P(X < 7.5) = \frac{7.5-5}{15-5} = \frac{2.5}{10} = 0.25 \Longrightarrow q_1 = 7.5$

$P(X < 12.5) = \frac{12.5-5}{15-5} = \frac{7.5}{10} = 0.275 \Longrightarrow q_3 = 12.5$

$iqr(X) = q_3 - q_1 = 12.5 - 7.5 = 5$

The standard deviation of X is $\sigma = \sqrt{225} = 15$. The ratio is $\frac{5}{15} = \frac{1}{3}$

(b) The quantile q_1 of Y is $qnorm(.25, 10, 15) = -0.1173463$

The quantile q_3 of Y is $qnorm(.75, 10, 15) = 20.11735$

$iqr(Y) = 20.11735 + 0.1173463 = 20.23469.$

The standard deviation of X is $\sigma = \sqrt{225} = 15$. The ratio is $\frac{20.23469}{15} = 1.34898$