S620 - Introduction To Statistical Theory - Homework 1

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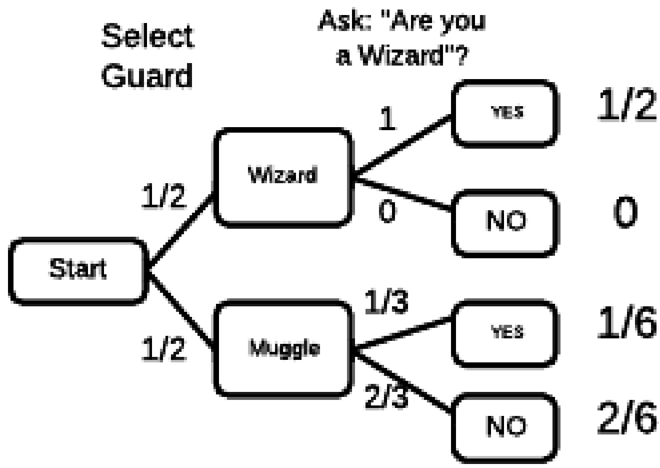
Formulate the scenarios described in Exercises 2.2, 2.3, and 2.4 as statistical decision problems.

(2.2)

1) Parameter Space: $\Theta = \{1, 2\} \subseteq \mathbb{R}$, where $\theta = 1 \in \Theta$ means that guard one is the wizard and $\theta = 2 \in \Theta$ means guard 2 is the wizard.
2) Sample Space: $\mathcal{X} = \{(1,1), (1,0), (2,1), (2,0)\} \subseteq \mathbb{R}^2$, where $x = (guard, response)$, and $guard \in \{1,2\}$ stands for the guard randomly chosen to be questioned ("are you the wizard?"); and *response* being 1 if the guard says he is the wizard and 0 otherwise.
3) Family of Probability Distributions: $\mathcal{P} = \{P_1, P_2\}$, where P_1 is the distribution in case guard 1 is the wizard and P_2 is the distribution in case guard 2 is the wizard. The distributions are given by:



For $P_1:$ we have: $P_1((1,1)) = \frac{1}{2}, P_1((1,0)) = 0, P_1((2,1)) = \frac{1}{6}, P_1((2,0)) = \frac{1}{3}$
For $P_2:$ we have: $P_1((1,1)) = \frac{1}{6}, P_1((1,0)) = \frac{1}{3}, P_1((2,1)) = \frac{1}{2}, P_1((2,0)) = 0$

4) Action Space: $\mathcal{A} = \{a_1, a_2\}$, where a_1 = ask guard 1 for directions and a_2 = ask guard 2 for directions.
5) Loss Function: By our setup we have: $\Theta \times \mathcal{A} = \{(1, a_1), (1, a_2), (2, a_1), (2, a_2)\}$. Therefore: $L((1, a_1)) = 0$ (no loss, we have asked the right guard), $L((1, a_2)) = 1000$ galleons (as determined in the statement problem, this is the loss associated with a failure to catch Hogwart's Express). Likewise, $L(2, a_1) = 1000$ and $L(2, a_2) = 0$. I am assuming that asking the wrong guard leads to failure of catching Hogwart's Express.
6) Decision Rules: $d : \mathcal{X} \to \mathcal{A}$, according to our setup we have the $4^2 = 16$ possibilities for non-randomized decision rule d: $d((1,1)) = a_1/a_2, d((1,0)) = a_1/a_2, d((2,1)) = a_1/a_2, d((2,0)) = a_1/a_2$, where a_1/a_2 means pick either a_1 or a_2, e.g., $d((1,1)) = a_2$ means that we ask guard 1 if he is a Wizard, he says yes, but choose to ask directions from guard 2.

(2.3)

1) Parameter Space: $\Theta = \{0, 1\} \subseteq \mathbb{R}$, where $\theta = 0 \in \Theta$ means no snow the next day and $\theta = 1 \in \Theta$ means there will be snow the next day.
2) Sample Space: $\mathcal{X} = \{(0,0), (0,1), (1,0), (1,1)\} \subseteq \mathbb{R}^2$. In particular, an element of the space is of the form $x = (r_1, r_2) \in \mathcal{X}$, where r_i = prediction of radio station i for $i = 1, 2$. The prediction is of the form $r_i = 1$ if radio station i predicts there will be snow the next day and 0 otherwise.
3) Family of Probability Distributions: $\mathcal{P} = \{P_0, P_1\}$, where P_0 is the distribution in case there will be no snow the next day, and P_1 is the distribution otherwise. The distributions are given by:

For $P_0:$ we have: $P_0((0,0)) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P_0((0,1)) = P_0((1,0)) = P_0((1,1))$; since each station information is i.i.d. and with equal probability $\frac{1}{2}$
For $P_1:$ we have: $P_1((0,0)) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}, P_1((0,1)) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}, P_1((1,0)) = \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}, P_1((1,1)) = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$

4) Action Space: $\mathcal{A} = \{\text{close school}, \text{open school}\}$

5) Loss Function: By our setup we have: $\Theta \times \mathcal{A} = \{(0, \text{close school}), (0, \text{open school}), (1, \text{close school}), (1, \text{open school})\}$, e.g., the tuple $(1, \text{open school})$ means the case when there is snow and the school remains open. Therefore: $L((0, \text{close school})) = c, L((0, \text{open school})) = 0, L((1, \text{close school})) = c, L((1, \text{open school})) = 2c$, where c is a constant representing the cost of the decision made.

6) Decision Rules: $d : \mathcal{X} \to \mathcal{A}$, according to our setup we have the $4^2 = 16$ possibilities for non-randomized decision rule d: $d((0,0)) = \text{close/open school}, d((0,1)) = \text{close/open school}, d((1,0)) = \text{close/open school}, d((1,1)) = \text{close/open school}$

(2.4)

1) Parameter Space: $\Theta = \{0, 1\} \subseteq \mathbb{R}$, where $\theta = 0 \in \Theta$ means the component is not functioning and $\theta = 1 \in \Theta$ means the component is functioning.

2) Sample Space: $\mathcal{X} = \{0, 1\} \subseteq \mathbb{R}^2$, where $x = 0 \in \mathcal{X}$ means the warning light if off and $x = 1 \in \mathcal{X}$ means the warning light goes on.

3) Family of Probability Distributions: $\mathcal{P} = \{P_0, P_1\}$, where P_0 is the distribution in case the component is not functioning, and P_1 in case the component is functioning. The distributions are given by:

For $P_0:$ we have: $P_0(0) = \frac{1}{3}, P_0(1) = \frac{2}{3}$.
For $P_1:$ we have: $P_1(0) = \frac{3}{4}, P_1(1) = \frac{1}{4}$.

4) Action Space: $\mathcal{A} = \{\text{stop launch}, \text{go on with launch}\}$

5) Loss Function: By our setup we have:
$\Theta \times \mathcal{A} = \{(0, \text{stop launch}),(0, \text{go on with launch}),(1, \text{stop launch}),(1, \text{go on with launch})\}$, e.g., the tuple $(1, \text{stop launch})$ represents the case when the component is functioning but the launch is stopped. Therefore, $L((0, \text{stop launch})) = 0, L((0, \text{go on with launch})) = 10\text{billion } \$, L((1, \text{stop launch})) = 5\text{billion } \$$, $L((1, \text{go on with launch})) = 0$

6) Decision Rules: $d : \mathcal{X} \to \mathcal{A}$, according to our setup we have the $2^2 = 4$ possibilities for non-randomized decision rule $d : d(0) = \text{stop launch/go on with launch}, d(1) = \text{stop launch/go on with launch}$, i.e., we can either stop or go on with the launch depending on the state of the warning light.