S620 - Introduction To Statistical Theory - Homework 5 - Bonus

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2. Suppose that $x_1, ..., x_n \sim Normal(\theta, 1)$ and consider the following two tests of $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$:

   (a) Reject H_0 iff $|\bar{x}| > k_1$, where the critical value k_1 is chosen so that the test has size α.

   (b) Reject H_0 iff $\sum_{i=1}^{n} x_i^2 > k_2$, where the critical value k_2 is chosen so that the test has size α.

   Is either of these tests uniformly more powerful than the other?

   Solution: Both of these test have the same size, so it makes sense to ask whether one of them is UMP than the other. To this end, let us compute the power of each of these tests:

   (a) Note that $\bar{x} \sim Normal(\theta, \frac{\sigma^2}{n}) = Normal(\theta, \frac{1}{n})$ since $\sigma = 1$. Let $\theta \in \Theta_1$, i.e., $\theta \neq 0$. Then the power is:

   $$P_\theta(\text{rejecting } H_0) = P_\theta(|\bar{x}| > k_1) = 1 - P_\theta(|\bar{x}| \leq k_1) = 1 - P_\theta(-k_1 \leq \bar{x} \leq k_1)$$

   Let $Z = \sqrt{n}(\bar{x} - \theta)$. Then $Z \sim Normal(0, 1)$. Then,

   $$1 - P_\theta(-k_1 \leq \bar{x} \leq k_1) = 1 - P_\theta(-(k_1 - \theta)\sqrt{n} \leq Z \leq (k_1 - \theta)\sqrt{n}) = 2 * pnorm(-(k_1 - \theta)\sqrt{n})$$

   This final expression valid for $\theta > 0$. If $\theta < 0$, then by symmetry take $|\theta|$ and you will get the same answer. This is just the two tails of the normal standard distribution where $z = -(k_1 - |\theta|)\sqrt{n}$. Note that the power of this test goes to zero as $|\theta|$ goes to infinity.

   (b) Let $\theta \in \Theta_1$ ($\theta \neq 0$). The power of test (b) is given by:

   $$P_\theta(\text{rejecting } H_0) = P_\theta\left(\sum_{i=1}^{n} x_i^2 > k_2\right) = 1 - P_\theta\left(\sum_{i=1}^{n} x_i^2 \leq k_2\right)$$

   At this point is apparent that this probability will depend on the choice of θ. Therefore, both tests will depend on θ, and no one is more powerful than the other.