S620 - Introduction To Statistical Theory - Homework 7

Enrique Areyan
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6.2 (ii) Find a minimal sufficient statistic for θ based on an independent sample of size n from the uniform distribution on $(\theta - 1, \theta + 1)$.

Solution: The pdf of a single uniform r.v. for the parameter θ is $f(X_1;\theta) = \dfrac{1}{\theta + 1 - (\theta - 1)} = \dfrac{1}{2}$, i.e.,

$$f(x_1;\theta) = \begin{cases} \frac{1}{2} & \text{if } \theta - 1 < x_1 < \theta + 1 \\ 0 & \text{otherwise} \end{cases}$$

The joint pdf for $X_1, \ldots, X_n \sim Uniform(\theta - 1, \theta + 1)$ is the product of the single pdfs, i.e., let $X = X_1, \ldots, X_n$

$$f(X;\theta) = \begin{cases} 2^{-n} & \text{if } \theta - 1 < x_1, x_2, \ldots, x_n < \theta + 1 \\ 0 & \text{otherwise} \end{cases}$$

This pdf can be written as follows:

$$f(X;\theta) = 2^{-n} \cdot \begin{cases} 1 & \text{if } \theta - 1 < min\{x_1, x_2, \ldots, x_n\} \text{ and } max(x_1, x_2, \ldots, x_n) < \theta + 1 \\ 0 & \text{otherwise} \end{cases}$$

Letting $h(x) = 2^{-n}$, $t(x) = (min\{x_1, x_2, \ldots, x_n\}, max\{x_1, x_2, \ldots, x_n\})$. Then $f(X;\theta) = h(x)g(t(x), \theta)$, and thus by the factorization theorem, $t(x)$ is a sufficient statistic. Moreover, $t(x)$ is also minimal. Let us check that this is the case by using Theorem 6.1

($\Rightarrow$) Suppose that $t(x) = t(y)$. Then, for any choice of θ_1, θ_2:

$$\Lambda_x(\theta_1, \theta_2) = \frac{f(X;\theta_1)}{f(X;\theta_2)} = \frac{h(x)g(t(x), \theta_1)}{h(x)g(t(x), \theta_2)} = \frac{g(t(x), \theta_1)}{g(t(x), \theta_2)} = \frac{g(t(y), \theta_1)}{g(t(y), \theta_2)} = \frac{h(y)g(t(y), \theta_1)}{h(y)g(t(y), \theta_2)} = \Lambda_y(\theta_1, \theta_2)$$

($\Leftarrow$) For this direction, let us prove the contrapositive. Suppose that $t(x) \neq t(y)$. We want to show that there exists θ_1, θ_2 such that $\Lambda_x(\theta_1, \theta_2) \neq \Lambda_y(\theta_1, \theta_2)$ for all θ_1, θ_2. Pick $\theta_2 = \theta_1 + 1$, then if $\Lambda_x(\theta_1, \theta_2) = 1$ it means that $g(t(x);\theta_1) = g(t(x);\theta_2) = 1$, but since $t(x) \neq t(y)$ we have that $min(X_i) \neq min(Y_i)$ and $max(X_i) \neq max(Y_i)$ and by our choice of θ_2 it follows that $g(t(x);\theta_1) = 0$ and so $\Lambda_y(\theta_1, \theta_2) = 0 \neq \Lambda_x(\theta_1, \theta_2)$

6.3 Independent factory-produced items are packed in boxes each containing k items. The probability that an item is in working order is $\theta, 0 < \theta < 1$. A sample of n boxes are chosen for testing, and X_i, the number of working items in the ith box, is noted. Thus $X_1, \ldots, X_n$ are a sample from a binomial distribution, Bin(k,θ), with index k and parameter θ. It is required to estimate the probability, θ^k, that all items in a box are in working order. Find the minimum variance unbiased estimator, justifying your answer.

Solution: Let us find the MVU estimator using Theorem 6.3. First, let us find a sufficient statistic for θ. The joint mass function is:

$$f(X;\theta) = f(X_1 = x_1, \ldots, X_n = x_n) = \binom{k}{x_1}\theta^{x_1}(1-\theta)^{k-x_1} \cdots \binom{k}{x_n}\theta^{x_n}(1-\theta)^{k-x_n} = \left\{\prod_{i=1}^{n}\binom{k}{x_i}\right\}\theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{nk-\sum_{i=1}^{n} x_i}$$

Letting $h(x) = \left\{\prod_{i=1}^{n}\binom{k}{x_i}\right\}$ and $t(x) = \sum_{i=1}^{n} x_i$, we find that $f(X;\theta) = h(x)g(t(x);\theta)$ so that, by the factorization theorem, $t(x)$ is a sufficient statistic. Now we need a sufficient statistic for θ^k. Consider the random variable I (indicator):

$$I(X_1) = \begin{cases} 1 & \text{if } X_1 = k \\ 0 & \text{otherwise} \end{cases}$$

Then, $E_\theta[I] = 1 \cdot Pr\{X_1 = k\} + 0 \cdot Pr\{X_1 \neq k\} = Pr\{X_1 = k\} = \binom{k}{k}\theta^k(1-\theta)^{k-k} = \theta^k$, so that I is an unbiased estimator for θ^k. Finally, by Theorem 6.3, the following estimator is the minimum variance unbiased estimator:

$$\chi(T) = E[I|T(X) = t] = E\left[I|\sum_{i=1}^{n} x_i = t\right] = 1 \cdot Pr\{I = 1|\sum_{i=1}^{n} x_i = t\} + 0 \cdot Pr\{I = 0|\sum_{i=1}^{n} x_i = t\} = Pr\{X_1 = k|\sum_{i=1}^{n} x_i = t\}$$

Hence, the distribution of $\chi(T)$ can be computed as follow:

$$\begin{aligned}
\chi(T=t) &= Pr\{X_1 = k | \sum_{i=1}^{n} x_i = t\} && \text{definition of } \chi(T) \text{ above} \\
&= \frac{Pr\{X_1 = k, \sum_{i=1}^{n} x_i = t\}}{Pr\{\sum_{i=1}^{n} x_i = t\}} && \text{conditional prob.} \\
&= \frac{Pr\{\sum_{i=1}^{n} x_i = t | X_1 = k\} Pr\{X_1 = k\}}{Pr\{\sum_{i=1}^{n} x_i = t\}} && \text{conditional prob.} \\
&= \frac{Pr\{\sum_{i=2}^{n} x_i = t-k\} Pr\{X_1 = k\}}{Pr\{\sum_{i=1}^{n} x_i = t\}} && \text{by the conditional prob. } (t \geq k)
\end{aligned}$$

Note that since $X_1, \ldots, X_n \sim Bin(k, \theta)$, we have that $\sum_{i=1}^{n} x_i \sim Bin(nk, \theta)$. Hence,

$$\chi(T=t) = \frac{Pr\{\sum_{i=2}^{n} x_i = t-k\} Pr\{X_1 = k\}}{Pr\{\sum_{i=1}^{n} x_i = t\}} = \frac{\binom{kn-k}{t-k}\theta^{t-k}(1-\theta)^{kn-k-(t-k)}\binom{kn}{k}\theta^{k}(1-\theta)^{k-k}}{\binom{kn}{t}\theta^{t}(1-\theta)^{kn-t}} = \frac{\binom{kn-k}{t-k}}{\binom{kn}{t}}$$

Hence, the UMV unbiased estimator is $\boxed{\chi(T) = \frac{\binom{kn-k}{t-k}}{\binom{kn}{t}}}$. To complete our reasoning, we need only to prove that $t(x)$ is complete:

Let $\theta \in (0,1)$ and g be a real function. Suppose that $E_\theta g(T) = 0$. Then, by the law of the unconscious statistician:

$$\sum_{i=0}^{nk} g(i) \cdot P_\theta(T = i) = 0$$

Since $0 < \theta < 1$ and $T \sim Bin(nk, \theta)$, it must be the case that $P_\theta(T = i) > 0$ for $0 \leq i \leq nk$. Hence, for the above equality to hold, we must have that: $g(i) = 0$ for all i, or equivalently, $Pr_\theta\{g(T) = 0\} = 1$ for all θ. Thus, T is complete.