Systems of Linear Equations
Reading: Lay 1.1

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1 Some Notes on Style

1.1 About these notes

Much of the course will be based on Lay. I recommend you do the readings. When these notes cover topics from Lay, they will try to explain things in a slightly different way. This way, when Lay describes things in a way that you find confusing, these notes may be helpful. Similarly, when these notes are confusing, perhaps Lay will be helpful!

1.2 About math

You may notice that the material involves a large number of definitions. Math is very similar to (for instance) computer programming: you have to spell out what you mean as carefully as you can.

2 Linear Equations

In this lecture, we introduce the idea of solving a **linear equation**. Here is Lay's definition of a linear equation.

Definition 2.1. A linear equation is an equation in the variables $x_1, x_2, \ldots, x_n$ that can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \tag{1}$$

for some numbers $a_1, a_2, \ldots, a_n$ and some number b.

When we say in Definition 2.1 that an equation "can be written in the form" (1), we mean that by adding or subtracting terms to both sides of the equation, you can produce something with the form (1).

Example 2.2. You have already seen linear equations in one variable x_1 before. For instance,

$$5x_1 + 2 = 12.$$

This is a linear equation because we can subtract 2 from both sides to place it in the form

$$5x_1 = 10.$$

Here $n = 1$, $a_1 = 5$ and $b = 10$.

Example 2.3. In this course, we will usually have $n > 1$. That is, our linear equations will generally be in many variables! For instance,

$$x_1 + 3x_2 - 27x_3 - 3 = 1 + 2x_2$$

is a linear equation. We can subtract $2x_2$ from both sides and add 3 to both sides to put it in the form

$$x_1 + x_2 - 27x_3 = 4.$$

Now we will describe what it means to "solve" a linear equation. For a single linear equation in the variables $x_1, \ldots, x_n$, we say a list of numbers $(s_1, \ldots, s_n)$ is a solution if, when we substitute s_1 for x_1, s_2 for x_2, and so on, we get a true statement.

Example 2.4. Consider the linear equation from Example 2.2. Then 2 is a solution to this equation. Why? Because when we insert 2 in place of x_1, we get

$$5 * 2 = 10,$$

which is a true statement.

Similarly, (3, 1, 0) is a solution to the linear equation of Example 2.3. You should check this!

In the real world, you will generally be dealing with a large number of linear equations in the same variables $x_1, \ldots, x_n$. In a real life problem, the goal is often to find a set of values which solve multiple linear equations simultaneously.

Definition 2.5. A **system of linear equations** or "linear system" is a collection of one or more linear equations in the same variables $x_1, \ldots, x_n$.

Definition 2.6. A **solution** of a system of linear equations in the variables $x_1, \ldots, x_n$ is a list of numbers $(s_1, \ldots, s_n)$ that makes each equation of the system a true statement when $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$ respectively. The **solution set** of a system of linear equations is the collection of all solutions of the system.

Example 2.7. The following is an example of a system of linear equations:

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + 4x_2 + x_3 &= 1 \end{aligned}$$

This is a system of linear equations in the variables x_1, x_2, x_3 (the absence of x_3 in the first equation just means that $a_3 = 0$). A solution to this system is given by $(1, 1, -5)$. Indeed, substituting into the first equation gives

$$1 + 2 * 1 = 3,$$

which is a true statement, and substituting into the second equation gives

$$2 + 4 - 5 = 1,$$

which is also true.

3 Solutions to linear systems

Some linear systems have no solutions. To take a very simple example, the system

$$\begin{aligned} x_1 - 1 &= 0 \\ x_1 + 5 &= 0. \end{aligned}$$

The first equation is only true when $x_1 = 1$, but then the second equation is false.

One question which will occupy much of our time is the following: when do linear systems have solutions, and how many? Lay gives the following answer:

Theorem 3.1. *A system of linear equations has either no solutions, infinitely many solutions, or exactly one solution.*

This might seem mysterious right now, but we will talk more about it in later lectures. Right now, it is just important to be aware of this fact. We will say that a system of linear equations is **inconsistent** if it has no solutions; otherwise, we say that it is **consistent**.

4 Solving linear systems

You may have some ideas already about how to find solutions to systems of linear equations. Lay has some discussion about how to solve them using graphs. Another way you might be familiar with is by "substitution." For instance. if we have the system

$$\begin{aligned} 1.5x_1 + 4x_2 &= 3 \\ x_1 - 6x_2 &= 0, \end{aligned}$$

you could rearrange the second equation to see that $x_1 = 6x_2$. Putting this information into the first equation gives

$$1.5 * 6x_2 + 4x_2 = 3,$$

or $x_2 = 13/3$ (and so $x_1 = 26$). This is not a good method, but we will describe a better one.

4.1 Matrix Notation

A simple way to encode the information in a linear system is via matrix notation. Given a linear system like

$$\begin{aligned} x_1 + 16x_2 &= 3 \\ 2x_2 &= 4, \end{aligned}$$

the **coefficient matrix** is the rectangular array formed by lining up the coefficients of each x_i in columns (for instance, the coefficients a_1 go in the first column, etc). For the above linear system, this looks like

$$\begin{bmatrix} 1 & 16 \\ 0 & 2 \end{bmatrix}.$$

The **augmented matrix** has an added column, the last or rightmost column, which contains the constants from the right side of the linear equations:

$$\begin{bmatrix} 1 & 16 & 3 \\ 0 & 2 & 4 \end{bmatrix}. \tag{2}$$

We will call a matrix $m \times n$ if it has m rows and n columns. For instance, the augmented matrix (2) is a 2×3 matrix.

4.2 Equivalent linear systems

We will say that two linear systems are **equivalent** if they have the same solution sets–that is, if they have all the same solutions. Our main strategy here for solving linear systems will be to replace them with equivalent systems which are easier to solve.

So we need to know how to produce equivalent linear systems. We will talk for the rest of the lecture about three basic ways to do this.

1. Adding a multiple of one equation to another equation;

2. Interchanging two equations;

3. Multiplying an equation by a nonzero constant factor.

Note that we perform these operations on both sides of the equation!

Theorem 4.1. *If we perform any of the three operations above on a linear system, we produce an equivalent linear system.*

The reason we introduced matrix notation is that we can easily keep track of the elementary row operations by using the augmented matrices. The three operations above correspond to what we call the **elementary row operations**.

Definition 4.2. The following operations on matrices are called elementary row operations:

1. Replacing one row of the matrix by the sum of itself and a multiple of another row;

2. Interchanging two rows;

3. Multiplying all entries in a row by a nonzero constant factor.

What Theorem 4.1 says is that if we perform elementary row operations on the augmented matrix of a linear system, *we get the augmented matrix of an equivalent linear system* (make sure you understand this!). Another thing to note (Lay discusses this also) is that elementary row operations are reversible–for instance, if I multiplied a row by some number $c \neq 0$, I can get back the original matrix by multiplying by $1/c$.

4.3 How to use equivalent linear systems

In the next lecture, we will describe a detailed algorithm for solving linear systems by finding simpler equivalent systems. One thing that Lay tries to do in this section, though, is to give some idea of how to do it by using a couple examples. You should read the three examples in Lay, then try to work through them on your own!

The basic idea is to use our first elementary row operation (adding a multiple of a row) to get all zeros in the first column of the matrix (except possibly in one row). Then we repeat with the second column, and so on, clearing out entries in the columns that correspond to coefficients. This gives us the augmented matrix of a linear system where each equation has more zero coefficients. Hopefully we get rows with few nonzero coefficients and can finish up using substitution or other elementary techniques from your calculus (and precalculus) knowledge.

Example 4.3. We will use elementary row operations to find the solution set of the system

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + x_2 &= 4. \end{aligned}$$

We first write the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}.$$

We want to "clear out the first column". We add -2 times the first row to the second row, producing the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \end{bmatrix}.$$

Next, we multiply the second row by $-1/3$ to get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2/3 \end{bmatrix}.$$

Finally, we add -2 times the second row to the first row:

$$\begin{bmatrix} 1 & 0 & 7/3 \\ 0 & 1 & -2/3 \end{bmatrix}.$$

Clearly the linear system has only one solution: $x_1 = 7/3, x_2 = -2/3$. You should plug these in to the original system and make sure they work.

One other thing this method is useful for is recognizing inconsistent linear systems. The hallmark of such a system is that, when we try to eliminate from columns, we eventually clear out "too much" and end up with a row which corresponds to an equation which cannot be satisfied, ever. For instance, take the linear system whose augmented matrix is

$$\begin{bmatrix} 3 & 9 & 3 \\ 1 & 2 & 1 \\ 1 & -2 & 5 \end{bmatrix}.$$

We first multiply the first row by $1/3$:

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 1 & -2 & 5 \end{bmatrix}.$$

adding -1 times the first row to each of the other two rows gives

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 0 \\ 0 & -5 & 4 \end{bmatrix}.$$

Now adding -5 times the second row to the third row gives us

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

The last row corresponds to the equation $0 = 4$, which is never satisfied. Therefore, our original linear system is inconsistent–in order for it to have a solution, the equation $0 = 4$ would have to be satisfied, which it clearly is not.