Echelon Forms
Reading: Lay 1.2

August 28, 2013

1 From last time

We didn't get through the last example from the previous notes during lecture, so I will go over that first. I will also talk about how the number of solutions to a linear system is either zero, one, or infinity (also in the last notes, also didn't get through in last lecture).

2 Row Echelon Form and RREF

Recall the three elementary row operations from the last lecture. We will say that two matrices that can be reached from each other via a sequence of elementary row operations are **row equivalent**. So our goal is to develop an algorithm to turn a general augmented matrix into a row equivalent matrix which is somehow "simpler". But first we have to describe exactly what we mean by "simple".

Definition 2.1. A matrix is in **row echelon form** if:

1. All nonzero rows are above any rows of all zeros (that is, any rows which are all zeros are at the bottom of the matrix);

2. Each **leading entry** (that is, the leftmost nonzero entry) of a row is in a column to the right of the row above it;

3. All entries in a column below a leading entry are zeros.

Example 2.2. The following matrices are in row echelon form:

$$\begin{bmatrix} 1 & 16 & 3 \\ 0 & 2 & 9 \end{bmatrix}, \quad \begin{bmatrix} 5 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

The following are not:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 5 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 1 & 0. \end{bmatrix}.$$

The name "echelon" really means "like a staircase." The rough picture is that there is some diagonal line going down and right across the matrix, and everything to the left of this line is zeros.

Definition 2.3. We say a matrix is in **reduced row echelon form** or **RREF** if it is in row echelon form and also has the following additional properties:

1. The leading entry in each nonzero row is 1;

2. Each leading 1 is the only nonzero entry in its column.

Example 2.4. The following matrix is in RREF:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The RREF will be the "simple form" that we will try to reduce every matrix to for ease of solution. In this lecture, we will describe a "row reduction algorithm" (sometimes called "Gaussian elimination") to do this. But before we describe the algorithm, let's mention two questions that we need the answer to. The first question: if we are given a matrix, can we always find a row-equivalent matrix which is in RREF? Second: can there be more than one row-equivalent matrix which is in RREF? These are answered by the following theorem:

Theorem 2.5. *Each matrix is row equivalent to one and only one reduced row echelon form matrix.*

The following definition is maybe a little tricky, so read carefully.

Definition 2.6. If A is a matrix and B is the matrix which is row-equivalent to A which is in RREF, we will say that B is the reduced row echelon form of A (or B is the RREF of A).

3 Row Reduction

3.1 The algorithm

The following definitions will seem somewhat strange at first, but it is useful shorthand for describing the algorithm.

Definition 3.1. Let B be a matrix. A **pivot position** is a location in B where there is a leading 1 in the RREF of B. A **pivot column** is a column that contains a pivot.

The row reduction algorithm is best described broken down into steps:

1. Begin with the leftmost column which does not consist of all zeros. This is a pivot column, and the pivot position will be at the top.

2. Select a nonzero entry in the pivot column and use the "exchange rows" operation (from our list of elementary row operations) to move this to the top row (the pivot position).

3. Add multiples of the row containing the pivot position to the other rows to produce all zeros below the pivot position.

4. Now ignore the row containing the pivot position–pretend it is not part of the matrix anymore. We repeat the process now on the **submatrix** (that is, part of a matrix) that is left when we ignore the pivot row. We keep repeating the process until there are no more columns to clear out.

   At this point, we have produced a matrix in row echelon form. The final step will turn it into a matrix in RREF.

5. Beginning with the rightmost pivot, create zeros above each pivot position using row operations. Finally, multiply each row by a constant so that there is a 1 in each pivot position. (We can in fact do these two

things in the opposite order, and sometimes this makes the calculation easier.)

This is a lot for you to try to remember in words. The easiest way to get the hang of the row reduction algorithm is to practice until you know it by heart. We will work an example so that you can see the process in action.

Example 3.2. We find the RREF of the matrix

$$\begin{bmatrix} 0 & 1 & 3 & 3 \\ 5 & 3 & 3 & 3 \\ 4 & 1 & 3 & 3 \end{bmatrix}.$$

First, note that there are nonzero entries in the left column. This will be the first pivot column. Swap the bottom and top rows:

$$\begin{bmatrix} 4 & 1 & 3 & 3 \\ 5 & 3 & 3 & 3 \\ 0 & 1 & 3 & 3 \end{bmatrix}.$$

The first pivot position is the top-left entry of the matrix. We add $-5/4$ times the first row to the second row to clear out the pivot column:

$$\begin{bmatrix} 4 & 1 & 3 & 3 \\ 0 & 7/4 & -1/4 & -1/4 \\ 0 & 1 & 3 & 3 \end{bmatrix}.$$

The next pivot column is the second column; the pivot position is the place in the matrix that is currently occupied by the number $7/4$. We add $-4/7$ times the second row to the third row to clear out the column below the pivot position.

$$\begin{bmatrix} 4 & 1 & 3 & 3 \\ 0 & 7/4 & -1/4 & -1/4 \\ 0 & 0 & 22/7 & 22/7 \end{bmatrix}.$$

This is now in a row echelon form. We now multiply the bottom row by $7/22$, the middle row by $4/7$, and the top row by $1/4$:

$$\begin{bmatrix} 1 & 1/4 & 3/4 & 3/4 \\ 0 & 1 & -1/7 & -1/7 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

We complete the RREF by clearing out the entries above the pivot positions, starting with the third column ("third" meaning "third from the left") and then the second column:

$$\begin{bmatrix} 1 & 1/4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and finally

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

. This is the RREF of our matrix.

3.2 Using RREF: parametrized families of solutions

Recall that every linear system either has zero solutions, one solution, or infinitely many solutions. We will discuss how to see which of these is true for a given linear system by looking at the RREF of the augmented matrix.

Consider the matrix from the last example. It is the augmented matrix of the linear system

$$\begin{aligned} x_2 + 3x_3 &= 3 \\ 5x_1 + 3x_2 + 3x_3 &= 3 \\ 4x_1 + x_2 + 3x_3 &= 3. \end{aligned}$$

What we just showed was that the solutions to this system are the same as the solutions to the equivalent linear system

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 1. \end{aligned}$$

So this is a case where our linear system has exactly one solution.

Suppose instead that we had a linear system whose augmented matrix had RREF

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This corresponds to the linear system

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 + x_3 &= 2. \end{aligned}$$

In a case like this, what you do is solve the equations for the variables in the pivot columns (Lay calls these **basic variables**):

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= 2 - x_3. \end{aligned}$$

We say that x_3 is a **free variable** because these equations do not restrict x_3; we can choose x_3 to be anything we want, and we can still find values for x_1 and x_2 which will make this a solution. In this case, every choice of x_3 gives a solution to the system, and similarly every solution to the system gives a value of x_3. For instance, $(-10, -3, 5)$ is a solution (check this). So is $(2, 3, -1)$. Clearly this system of equations has infinitely many solutions!

As mentioned above, in this case every choice of the free variable gives a solution to the system. In this case, we say that the family of solutions is **parametrized** by x_3. We could have of course solved the equations for different variables and parametrized the solutions differently. However, the convention in linear algebra is usually to have the free variables be the ones which do not correspond to the pivot columns.

3.3 Existence and uniqueness

We have seen now three types of examples: inconsistent systems (which have no solution), as well as the two types of consistent systems (where you have either one solution or a parametrized family of infinitely many).

If you look at the example from the last set of notes, you saw that the reason the example there was an inconsistent system was that it was equivalent to a system which contained an equation of the form $0 = 4$. In general, you can tell whether a system is inconsistent by the fact that its RREF has a row of the form:

$$\begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}.$$

Clearly this corresponds to a linear equation of the form $0 = 1$, which is never satisfied.

We can state everything we have learned about the RREF and solutions in the following theorem:

Theorem 3.3. *A linear system is consistent–that is, has at least one solution–if and only if the rightmost column of the augmented matrix A is not a pivot column. What this means is that the linear system is consistent if and only if there is a matrix B such that*

- *B is row-equivalent to A,*
- *B is in row echelon form,*
- *and B has no row of the form*

$$\begin{bmatrix} 0 & 0 & \dots & b \end{bmatrix}$$

for some $b \neq 0$.

If the system is consistent, then it either has one solution (if there are no free variables) or infinitely many (if there is at least one free variable).