Eigenvalues / Characteristic Equation
Lay 5.2

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Today's topic is mostly how to find the eigenvalues of a matrix (given the eigenvalues, we learned last time how to find the eigenspaces).

1 The characteristic equation

Let A be an $n \times n$ matrix. A scalar λ is an eigenvalue of A if and only if there is some nonzero $\mathbf{x}$ such that

$A\mathbf{x} = \lambda\mathbf{x}$, or equivalently $(A - \lambda I)\mathbf{x} = \mathbf{0}$.Therefore, λ is an eigenvalue if and only if $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, which (by the invertible matrix theorem) happens if and only if $(A - \lambda I)$ is not invertible. But this is equivalent to $\det(A - \lambda I) = 0$.

We have just proved the following theorem:

Theorem 1.1. *A scalar λ is an eigenvalue of A if and only if it satisfies the* **characteristic equation**

$$\det(A - \lambda I) = 0.$$

Example 1.2. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Notice that this matrix is upper triangular, so our previous theorem gives us that the eigenvalues are the diagonal entries $3, 2$. We can also use the characteristic equation:

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 1 & 1 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (3-\lambda)(2-\lambda)(2-\lambda) = 0,$$

which gives the roots $\lambda = 3, 2$, and 1. Finding the eigenvectors corresponding to 2:

$$A - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

and augmenting and finding the RREF gives us that the eigenspace is spanned by

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The eigenspace corresponding to the eigenvalue 3 is spanned by the vector $(1, 0, 0)$ (I leave this to you to check).

Proposition 1.3. *Given an $n \times n$ matrix A, the function $\det(A - \lambda I)$ (regarded as a function of λ) is a polynomial of degree n. We call it the* **characteristic polynomial**.

Since the characteristic equation has the form polynomial $= 0$, an eigenvalue can appear multiple times as a root of the characteristic equation (for instance, the equation $(\lambda - 1)^2 = 0$ has 1 as a double root).

Definition 1.4. An eigenvalue λ of the matrix A has **algebraic multiplicity** k if its multiplicity as a root of the characteristic equation is k. So a double root of the characteristic equation has algebraic multiplicity 2, etc.

Example 1.5. A 10×10 matrix A has characteristic polynomial

$$\lambda^{10} - \lambda^8.$$

What are its eigenvalues and their algebraic multiplicities?

We can factor the polynomial to get $\lambda^8(\lambda^2 - 1) = \lambda^8(\lambda + 1)(\lambda - 1)$. So there are three eigenvalues: $0, 1$, and -1. The eigenvalue 0 has algebraic multiplicity 8 and the others have algebraic multiplicity 1.

2 Similarity

Definition 2.1. Two $n \times n$ matrices A and B are **similar** if there is some invertible matrix P such that

$$B = P^{-1}AP. \tag{1}$$

Note that the definition is symmetric: if there is matrix such that (1) holds, then multiplying on the left by P and on the right by P^{-1} gives $A = PBP^{-1}$, and since P^{-1} is invertible, we see that the same condition holds.

Similarity is important because two matrices that are similar represent *the same linear transformation after a change of basis.* We will return to this point soon. Right now, we will focus on one aspect of this fact:

Theorem 2.2. *If A and B are similar, then they have the same characteristic polynomial and eigenvalues.*

Proof.

$$\begin{aligned}\det(B - \lambda I) &= \det(P^{-1}AP - \lambda P^{-1}P) \\ &= \det(P^{-1}(A - \lambda I)P) \\ &= \det(P^{-1})\det(A - \lambda I)\det(P) \\ &= \det(A - \lambda I),\end{aligned}$$

since $1 = \det(I) = \det(PP^{-1}) = \det(P)\det(P^{-1})$.

□

However, if A and B have the same eigenvalues, they are not necessarily similar, so there is no converse to the above theorem in general.

3 An application and motivation for what comes next

If the eigenvectors of A form a basis for $\mathbb{R}^n$, then a lot of problems are made easier.

Say we have a sequence (a **dynamical system**, to use the term Lay likes) of vectors $\{\mathbf{x}_k\}$ such that

$\mathbf{x}_{k+1} = A\mathbf{x}_k = A^{k+1}\mathbf{x}_0$ for all k (think back to Markov chains for a special class of dynamical systems; note that most dynamical systems are not Markov chains). If the eigenvectors of A form a basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ (with eigenvalues $\lambda_1, \ldots, \lambda_n$) for $\mathbb{R}^n$, then we can write $\mathbf{x}_0 = c_1\mathbf{b}_1 + \ldots c_n\mathbf{b}_n$, so

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{b}_1 + \ldots + c_n\lambda_n^k\mathbf{b}_n.$$

3.1 Fibonacci numbers

Here is an application to show why this is useful. The Fibonacci numbers are a sequence of numbers defined by

$$F_0 = 0, F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \text{ for larger } n.$$

This sequence describes the number of breeding pairs of rabbits at generation n in a simple model for rabbit breeding.

The short description: we start with one pair of rabbits ($F_1 = 1$), they mate and undergo one month of pregnancy and so the number of rabbits is unchanged ($F_2 = 1$), then the third month the female gives birth and we have 2 pairs ($F_3 = 2$), and in general:

- The number of pairs of rabbits alive in month n is equal to the number of pairs of rabbits alive in month $n - 1$ plus the number of rabbits alive at time $n - 2$.

It is a simply defined sequence of numbers and the model is intuitively interesting. Yet, it is not clear how to find the number of rabbits alive at time n without doing a lot of tedious adding. We will show how to get around this problem using eigenvectors.

Defining the vectors

$$\mathbf{x}_n = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix},$$

then for $n \geq 1$, the vectors $\{\mathbf{x}_n\}$ obey the equation $A\mathbf{x}_{n+1} = Ax_n$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is not hard to show that the eigenvalues of this matrix are

$$\lambda_1 = \frac{1}{2}(1 + \sqrt{5}), \quad \lambda_2 = \frac{1}{2}(1 - \sqrt{5}).$$

Since there are two eigenvalues, the eigenvectors of A form a basis for $\mathbb{R}^2$ (why?). The eigenspace corresponding to λ_1 is spanned by

$$\mathbf{v}_1 = \begin{bmatrix} (1/2)(1 + \sqrt{5}) \\ 1 \end{bmatrix}$$

and the eigenspace corresponding to λ_2 is spanned by

$$\mathbf{v}_2 = \begin{bmatrix} (1/2)(1 - \sqrt{5}) \\ 1 \end{bmatrix}.$$

Since $\mathbf{x}_1 = (1, 0)$, we can write $\mathbf{x}_1 = (\mathbf{v}_1 - \mathbf{v}_2)/\sqrt{5}$.
Therefore

$$\mathbf{x}_k = \frac{\lambda_1^{k-1}\mathbf{v}_1 - \lambda_2^{k-1}\mathbf{v}_2}{\sqrt{5}},$$

and so the nth Fibonacci number is given by

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

It is quite a surprise that although the Fibonacci numbers are all integers, the number $\sqrt{5}$ is so important in finding them!