Homework 8 Solutions

1. Consider the problem of testing whether a DFA and a regular expression are equivalent. Express this problem as a language and show that it is decidable.

   Answer: Define the language as

   $C = \{\langle M, R\rangle \mid M \text{ is a DFA and } R \text{ is a regular expression with } L(M) = L(R)\,\}.$

   Recall that the proof of Theorem 4.5 defines a Turing machine F that decides the language $EQ_{\text{DFA}} = \{\, \langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\,\}$. Then the following Turing machine T decides C:

   T = "On input $\langle M, R\rangle$, where M is a DFA and R is a regular expression:
   1. Convert R into a DFA D_R using the algorithm in the proof of Kleene's Theorem.
   2. Run TM F from Theorem 4.5 on input $\langle M, D_R\rangle$.
   3. If F accepts, *accept*. If F rejects, *reject*."

2. Let $A\varepsilon_{\text{CFG}} = \{\, \langle G\rangle \mid G \text{ is a CFG that generates } \varepsilon\,\}$. Show that $A\varepsilon_{\text{CFG}}$ is decidable.

   Answer: We need to ensure that we test all derivations, but we also need the derivations not to be infinite, or to loop forever. To do this, we first convert the CFG G into an equivalent CFG $G' = (V, \Sigma, R, S)$ in Chomsky normal form. If $S \to \varepsilon$ is a rule in G', where S is the start variable, then clearly G' generates ε, so G also generates ε since $L(G) = L(G')$. Since G' is in Chomsky normal form, the only possible ε-rule in G' is $S \to \varepsilon$, so the only way we can have $\varepsilon \in L(G')$ is if G' includes the rule $S \to \varepsilon$ in R. Hence, if G' does not include the rule $S \to \varepsilon$, then $\varepsilon \notin L(G')$. Thus, a Turing machine that decides $A\varepsilon_{\text{CFG}}$ is as follows:

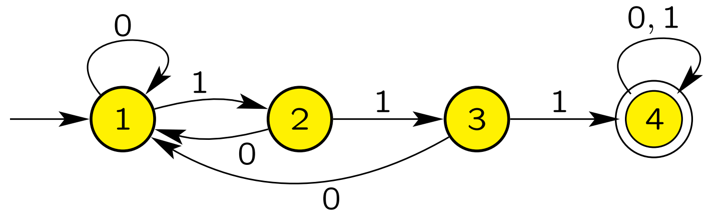
   M = "On input $\langle G\rangle$, where G is a CFG:
   1. Convert G into an equivalent CFG $G' = (V, \Sigma, R, S)$ in Chomsky normal form.
   2. If G' includes the rule $S \to \varepsilon$, *accept*. Otherwise, *reject*."

3. Let $\Sigma = \{0, 1\}$, and define

$$A = \{ \langle R \rangle \mid R \text{ is a regular expression describing a language over } \Sigma \text{ containing at least one string } w \text{ that has } 111 \text{ as a substring (i.e., } w = x111y \text{ for some } x \text{ and } y) \}.$$

Show that A is decidable.

Answer: Define the language $C = \{ w \in \Sigma^* \mid w \text{ has } 111 \text{ as a substring} \}$. Note that C is a regular language with regular expression $(0 \cup 1)^* 111 (0 \cup 1)^*$ and is recognized by the following DFA D_C:



Now consider any regular expression R with alphabet Σ. If $L(R) \cap C \neq \emptyset$, then R generates a string having 111 as a substring, so $\langle R \rangle \in A$. Also, if $L(R) \cap C = \emptyset$, then R does not generate any string having 111 as a substring, so $\langle R \rangle \notin A$. By Kleene's Theorem, since $L(R)$ is described by regular expression R, $L(R)$ must be a regular language. Since C and $L(R)$ are regular languages, $C \cap L(R)$ is regular since the class of regular languages is closed under intersection, as was shown in an earlier homework. Thus, $C \cap L(R)$ has some DFA $D_{C \cap L(R)}$. Theorem 4.4 shows that $E_{\text{DFA}} = \{ \langle B \rangle \mid B \text{ is a DFA with } L(B) = \emptyset \}$ is decidable, so there is a Turing machine H that decides E_{DFA}. We apply TM H to $\langle D_{C \cap L(R)} \rangle$ to determine if $C \cap L(R) = \emptyset$. Putting this all together gives us the following Turing machine T to decide A:

T = "On input $\langle R \rangle$, where R is a regular expression:

1. Convert R into a DFA D_R using the algorithm in the proof of Kleene's Theorem.

2. Construct a DFA $D_{C \cap L(R)}$ for language $C \cap L(R)$ from the DFAs D_C and D_R.

3. Run TM H that decides E_{DFA} on input $\langle D_{C \cap L(R)} \rangle$.

4. If H accepts, *reject*. If H rejects, *accept*."

4. Consider the emptiness problem for Turing machines:

$$E_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a Turing machine with } L(M) = \emptyset \}.$$

Show that E_{TM} is co-Turing-recognizable. (A language L is co-Turing-recognizable if its complement $\overline{L}$ is Turing-recognizable.) Note that the complement of E_{TM} is

$$\overline{E_{\text{TM}}} = \{ \langle M \rangle \mid M \text{ is a Turing machine with } L(M) \neq \emptyset \}.$$

(Actually, $\overline{E_{\text{TM}}}$ also contains all $\langle M \rangle$ such that $\langle M \rangle$ is not a valid Turing-machine encoding, but we will ignore this technicality.)

Answer: We need to show there is a Turing machine that recognizes $\overline{E_{\text{TM}}}$, the complement of E_{TM}. Let $s_1, s_2, s_3, \ldots$ be a list of all strings in Σ^*. For a given Turing machine M, we want to determine if any of the strings $s_1, s_2, s_3, \ldots$ is accepted by M. If M accepts at least one string s_i, then $L(M) \neq \emptyset$, so $\langle M \rangle \in \overline{E_{\text{TM}}}$; if M accepts none of the strings, then $L(M) = \emptyset$, so $\langle M \rangle \notin \overline{E_{\text{TM}}}$. However, we cannot just run M sequentially on the strings $s_1, s_2, s_3, \ldots$. For example, suppose M accepts s_2 but loops on s_1. Since M accepts s_2, we have that $\langle M \rangle \in \overline{E_{\text{TM}}}$. But if we run M sequentially on $s_1, s_2, s_3, \ldots$, we never get past the first string. The following Turing machine avoids this problem and recognizes $\overline{E_{\text{TM}}}$:

R = "On input $\langle M \rangle$, where M is a Turing machine:
1. Repeat the following for $i = 1, 2, 3, \ldots$.
2. Run M for i steps on each input $s_1, s_2, \ldots, s_i$.
3. If any computation accepts, *accept*.

5. Let A and B be two disjoint languages over a common alphabet Σ. Say that language C ***separates*** A and B if $A \subseteq C$ and $B \subseteq \overline{C}$. Show that if A and B are any two disjoint co-Turing-recognizable languages, then there exists a decidable language C that separates A and B. (A language L is co-Turing-recognizable if its complement $\overline{L}$ is Turing-recognizable.)

Answer: Suppose that A and B are disjoint co-Turing-recognizable languages. We now prove that there exists a decidable language C that separates A and B. Since A is co-Turing-recognizable, its complement $\overline{A}$ must have an enumerator $E_{\overline{A}}$. Similarly, the fact that B is co-Turing-recognizable implies $\overline{B}$ has an enumerator $E_{\overline{B}}$. Since A and B are disjoint, i.e., $A \cap B = \emptyset$, we have that $\overline{A} \cup \overline{B} = \Sigma^*$ by DeMorgan's law. Thus, every string in Σ^* is in the union of $\overline{A}$ and $\overline{B}$. Furthermore, since A and B are disjoint, every string in B is in $\overline{A}$, and every string in A is in $\overline{B}$.

Using these facts, we construct a Turing machine M as follows:

M = "On input w, where $w \in \Sigma^*$:
1. Run $E_{\overline{B}}$ and $E_{\overline{A}}$ in parallel.
2. Alternating between the enumerators, and starting with $E_{\overline{B}}$, compare the outputs of each of the enumerators, one string at a time, to the input w.
3. If some output of $E_{\overline{B}}$ matches w, *accept*.
If some output of $E_{\overline{A}}$ matches w, *reject*."

Let C be the language recognized by TM M. Since $\overline{A} \cup \overline{B} = \Sigma^*$, every string is enumerated by $E_{\overline{A}}$ or $E_{\overline{B}}$ (or both). Hence, M will halt on all inputs, so M is a decider for language C.

We now need to show that C separates A and B. Since every string in A is in $\overline{B}$, the output of $E_{\overline{B}}$ contains all strings of A. Thus, M accepts all strings that are output by only $E_{\overline{B}}$, so M accepts all strings of A since $E_{\overline{A}}$ never outputs any strings in A. Likewise, since every string in B is in $\overline{A}$, the output of $E_{\overline{A}}$ contains all strings of B. But M rejects all strings that are output by only $E_{\overline{A}}$, so M rejects all strings in B since $E_{\overline{B}}$ never outputs strings from B. Thus, M accepts all strings in A and rejects all strings in B, so its language C separates A and B.

Note that we did not prove which set C of strings M accepted. The particular language of C depends on the order of the outputs of the enumerators. However, the only strings in question are the strings that are in $\overline{A} \cap \overline{B}$. Whether these strings are in C or in $\overline{C}$ is not relevant to the question of separating A and B.