609 Homework 2

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(1.13) Double Counting: Let us count in two different ways the number of edges in a complete graph with n vertices.

(i) We need to have an edge between every pair of vertices. Each edge is a 2- subset of a set with n elements, by definition there are $\binom{n}{2}$ edges.

(ii) Let $0 \leq k \leq n$. To count the number of edges in a complete graph on n vertices we could count edges from disjoint sets of vertices as follow:

(1) Take a set of k out of the n vertices. Form a complete k graph, for which we need $\binom{k}{2}$ edges.

(2) Form the complete graph out of the remaining $n-k$ vertices. We will need $\binom{n-k}{2}$ edges.

(3) Connect the vertices of the k set with the vertices of the $n-k$ set. You will need $k(n-k)$ edges.

Since edges in (1),(2) and (3) are disjoint and together completely connect the graph, we will need $\binom{k}{2} + k(n-k) + \binom{n-k}{2}$ many edges to completely connect n vertices.

By the Double Counting Principle, the above counts (i) and (ii) must agree, hence:

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$$

(1.21) Let $k \geq 2n$. Arrange the k sweets in a linear fashion: $1, 2, ..., k$. The first child starts to pick at position 1, but she has to have two candies. Hence, the first two positions are unavailable for the second child. Likewise, the last child can pick any of the remaining positions except for the last position since she has to have at least two candies. Thus far we have 3 positions unavailable.

Now, child two may start to pick at position 3. I she does, then child 4 cannot start to pick at position 4 since each child two has to have at least two. Hence, one more position is unavailable. This pattern repeats for children 2 through $n-1$. Hence, for each of these $n-2$ children we have to remove one position.

In conclusion, there are $k-3-(n-2) = k-n-1$ positions for the children to choose from. Since the first child always starts at position one, there will be $n-1$ children choosing these positions. Hence, there are:

$$\binom{k-n-1}{n-1}$$

ways to distribute k sweets to n children if each child is suppose to get at least two of them.

(1.32) Let X be the set of all functions $f : [m] \to [n]$, where $m \geq n$ and let $A_i = \{f : f(j) \neq i \text{ for all } i\}$. Then $\bigcup_i^n A_i$ is the set of all functions that are not onto, which means that the complement $X - \bigcup_i^n A_i$ is the set of all onto functions. We wish to compute the cardinality of this set, i.e.,

$$|X - \bigcup_i^n A_i| = \sum_{I \subseteq [n]} (-1)^{|I|} |A_I| \quad \text{By the } \underline{\text{Inclusion Exclusion Principle}}$$

We know that $|A_i| = (n-1)^m$, counting all the functions from $[m]$ to $[n-1]$. But then, $|A_I| = (n-|I|)^m$. Hence,

$$\begin{aligned}
\sum_{I \subseteq [n]} (-1)^{|I|} |A_I| &= \sum_{I \subseteq [n]} (-1)^{|I|} (n-|I|)^m && \text{Replacing } |A_I| = (n-|I|)^m \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m && \text{Replacing } i = |I| \\
&= \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^m && \text{Since } (n-n)^m = 0 \text{, so the last term does not contribute}
\end{aligned}$$

(1.33) Consider the universe $X = X_{n,k}$ of all integer solutions to the equation $x_1 +_2 +... + x_n = k$ with all $x_i \geq 0$. Let A_i be the set of solutions with $x_i \geq l$. We wish to compute the cardilanitly of $X - \bigcup_{i}^{n} A_i$, i.e., the set of solutions with all $a \leq x_i < l$.

$$|X - \bigcup_{i}^{n} A_i| = \sum_{I \subseteq [n]} (-1)^{|I|}|A_I| \quad \text{By the \underline{Inclusion Exclusion Principle}}$$

By proposition (1.5), we know that $|X_{n,k}| = \binom{n+k-1}{k}$. Hence, we can conclude that $|A_i| = |X_{n,k-l}| = \binom{n+k-l-1}{k-l}$ which means that $|A_I| = \binom{n+k-|I|l-1}{k-|I|l}$. Replacing this into our IEP equation:

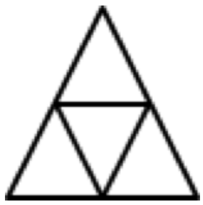
$$\begin{aligned}
\sum_{I \subseteq [n]} (-1)^{|I|}|A_I| &= \sum_{I \subseteq [n]} (-1)^{|I|}\binom{n+k-|I|l-1}{k-|I|l} && \text{Replacing } |A_I| = \binom{n+k-|I|l-1}{k-|I|l} \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i}\binom{n+k-il-1}{k-il} && \text{Replacing } i = |I|
\end{aligned}$$

(1.36) Let $I \subseteq J$. Consider $\sum_{I \subseteq K \subseteq J} (-1)^{|K \backslash I|}$. If we let $i = |K \backslash I|$, then we have to sum over $|J \backslash I|$ considering each possible subset form by $K \cup |J \backslash I|$ (completing K with elements of J) as follow:

$$\begin{aligned}
\sum_{I \subseteq K \subseteq J} (-1)^{|K \backslash I|} &= \sum_{i=0}^{|J \backslash I|} (-1)^i \binom{|J \backslash I|}{i} && \text{Substituting variables, we have to sum over } |J \backslash I| \\
&= \sum_{i=0}^{|J \backslash I|} 1^{|J \backslash I|-i}(-1)^i \binom{|J \backslash I|}{i} && \text{Since 1 to any power is just 1} \\
&= (1-1)^{|J \backslash I|} && \text{By the Binomial Theorem} \\
&= 0^{|J \backslash I|}
\end{aligned}$$

Now, if $J = I$ then $|J \backslash I| = \emptyset$ and hence $0^{|J \backslash I|} = 0^0 = 1$. Otherwise, if $I \subset J$ then $|J \backslash I| = n$ for some n and hence $0^n = 0$.

(4.1) Divide the equilateral triangle of length 1 into four equilateral triangles of length $\frac{1}{2}$ each, as follow:



Any two points that we pick in a equilateral triangle of length $\frac{1}{2}$ are going to have distance apart at most $\frac{1}{2}$. Indeed, this maximum distance is achieve between points at different vertices of the triangle. Any other points that we pick are going to have distance less than $\frac{1}{2}$.

Now, if we pick 5 points on the triangle of length 1, then it must be the case that these points are in some of the four sub triangles. Since there are 4 sub triangles and 5 points, by the Pigeonhole Principle, there must be at least two points on the same region. These two points will have distance at most $\frac{1}{2}$, showing the result.

(4.4) Consider the two mappings: $m1$: $i \mapsto (2i, 2i-1)$ and $m2$:$i \mapsto (i, 2n-i+1)$, where $i = 1, .., n$. Note that each mapping cover all numbers from $1, ..., 2n$. Now choose $n+1$ distinct integers from the set $\{1, 2, ..., 2n\}$. For each mapping, we have n pigeonholes, one for each i. Since we are choosing $n+1$ pigeons, by the Pigeonhole Principle, there is at least one pigeonhole that contains two pigeons.

In terms of the first mapping, this means that there exists an i such that two out of the $n+1$ numbers chosen, say a_i, a_j are such that $i \mapsto (2i, 2-1) = (a_i, a_j)$, and hence $a_i + 1 = a_j$ (a_j is consecutive to a_i).
In terms of the second mapping, two of the numbers chosen, a_i, a_j, are such that $i \mapsto (i, 2n-i+1) = (a_i, a_j)$, and hence $a_i + a_j = 2n+1$.

(4.5) Every set of $n+1$ distinct integers chosen from $\{1, 2, ..., 2n\}$ contains two numbers such that one divides the other. To see why this is the case, write every number x in the form $x = k_k 2^a$, where k_x is an odd number between 1 and $2n-1$. Take odd pigeonholes $1, 3, 5, ..., 2n-1$ and put x into the pigeonhole k_x. Since there are $n+1$ numbers to put into n holes, by the PHP some hole must have two numbers $x < y \iff k_x 2^{a_1} < k_y 2^{a_2}$. But, since the numbers are in the same hole by definition $k_x = k_y = k \Rightarrow k2^{a_1} < k2^{a_2} \iff 2^{a_1} < 2^{a_2} \iff a_2 > a_1$. Hence, $x|y$ since $k2^{a_2} = k2^{a_1} \cdot 2^p$, where $p + a_1 = a_2$

(4.7) Suppose that n is a multiple of k. Let us construct a graph without $(k+1)$-cliques, in which the number of edges achieves the upper bound given by Turán's Theorem. Before we begin, let us define: n number of vertices, k number of clusters, i.e., clusters of vertices, $\frac{n}{k}$ is number of vertices per cluster.

To achieve Turán's upper bound, we are going to begin by splitting the n vertices into k equal size parts and join all pairs of vertices from different parts, thus forming a complete k-partite graph. Hence, there are $\frac{n}{k}(n - \frac{n}{k})$ edges between the first cluster and the second cluster. Now we need to count the number of edges between the second and third cluster, there are $\frac{n}{k}(n - 2\frac{n}{k})$ many edges (we take away 2 groups of vertices since we already counted the first and we nee not count the second). This patterns repeats for counting edges between cluster i and $i+1$: $\frac{n}{k}(n - i\frac{n}{k})$.

Since these are all disjoint edges, we sum the total of all clusters, i.e.: $|E| = \sum_{i=1}^{k} \frac{n}{k}(n - i\frac{n}{k})$. But:

$$
\begin{aligned}
\sum_{i=1}^{k} \tfrac{n}{k}(n - i\tfrac{n}{k}) &= \frac{n}{k}\sum_{i=1}^{k}(n - i\frac{n}{k}) && \text{Since } \tfrac{n}{k} \text{ does not depend on } i \\
&= \frac{n^2}{k}\sum_{i=1}^{k}(1 - \frac{i}{k}) && \text{Factoring } n \text{ out of the sum} \\
&= \frac{n^2}{k}(\sum_{i=1}^{k} 1 - \frac{1}{k}\sum_{i=1}^{k} i) && \text{Separating the sum} \\
&= \frac{n^2}{k}(k - \frac{k(k+1)}{2k}) && \text{Sum of constant and sum of first } k \text{ natural numbers.} \\
&= \frac{n^2}{k}\frac{2k^2 - k^2 - k}{2k} && \text{Summing fractions} \\
&= \frac{n^2}{k}\frac{k(k-1)}{2k} && \text{Grouping terms} \\
&= \frac{n^2}{k}\frac{(k-1)}{2} && \text{Cancelling k's} \\
&= \frac{n^2 k - n^2}{2k} && \text{Taking product on the numerator} \\
&= \frac{n^2}{2} - \frac{n^2}{2k} && \text{Separating fractions} \\
&= (1 - \frac{1}{k})\frac{n^2}{2} && \text{Common factor } \tfrac{n^2}{2} \\
&= |E|
\end{aligned}
$$

Which shows that the number of edges achieves the upper bound set by Turán's theorem.

(4.10) Given a sequence $A = (a_1, ..., a_n)$ of $n \geq rs + 1$ different real numbers, define the partial order $\preceq$ on A by:

$$a_i \preceq a_j \iff a_i \leq a_j \text{ and } i \leq j$$

Check that indeed this is a partial order:

Reflexivity: Let $a_i \in A$ for some $1 \leq i \leq n$. Since $a_i \leq a_i$ and $i \leq i$, it follows that $a_i \preceq a_i$.

Antisymmetry: Let $a_i, a_j \in A$ be such that $a_i \preceq a_j$ and $a_j \preceq a_i$. Then by definition of $\preceq$, $a_i \leq a_j$ and $a_j \leq a_i$ which means that $a_i = a_j$. Also, $i \leq j$ and $j \leq i$, hence $i = j$, so $a_i = a_j$.

Transitivity: Let $a_i, a_j, a_k \in A$. Suppose that $a_i \preceq a_j$ and $a_j \preceq a_k$. Then, $a_i \leq a_j$ and $a_j \leq a_k$, which means that $a_i \leq a_k$. Also, $i \leq j$ and $j \leq k$, so $i \leq k$. Since $a_i \leq a_k$ and $i \leq k$, it follows that $a_i \preceq a_k$

This shows that $\preceq$ is a partial order. Since by $|A| = n \geq rs + 1$, we can apply Lemma 4.6 (Dilworth) and conclude that there exists a chain of length $s+1$ or an antichain of length $r+1$. A chain in A of length $s+1$ is set $\{a_k, a_{k+1}, ..., a_{k+s+1}\}$ so that the elements form a total order, i.e.,

$$a_k \preceq a_{k+1} \preceq ... \preceq a_{k+s+1} \iff a_k \leq a_{k+1} \leq ... \leq a_{k+s+1} \text{ and } k \leq k+1 \leq ... \leq k+s+1$$

But since all $a_i s$ are different, we obtain that

$$a_k < a_{k+1} < ... < a_{k+s+1} \text{ and } k < k+1 < ... < k+s+1$$

This is by definition an increasing subsequence of $s+1$ terms.

An antichain in A of length $r+1$ is set $\{a_l, a_{l+1}, ..., a_{l+r+1}\}$ so that no two distinct elements are comparable:

$$a_l \not\preceq a_{l+1} \not\preceq ... \not\preceq a_{l+r+1} \iff a_l > a_{l+1} > ... > a_{l+r+1} \text{ and } l > l+1 > ... > l+r+1$$

This is by definition a decreasing subsequence of $s+1$ terms, which shows the result.

(4.11) Define the partial order $\preceq$ over the given n^2+1 points in $\mathbb{R}^2$ as follow:

$$(x, y) \preceq (z, w) \iff x \leq z \text{ and } y \leq w$$

It is easy to see that this is indeed a partial order. Since there are n^2+1 points, by Dilworth's Lemma, we can conclude that there exists a chain of $n+1$ elements or an anti chain of $n+1$ elements.

A chain of length $n+1$ is set $\{(x_k, y_k), (x_{k+1}, y_{k+1}), ..., (x_{k+n+1}, y_{k+n+1})\}$ so that the elements form a total order, i.e.,

$$(x_k, y_k) \preceq (x_{k+1}, y_{k+1}) \preceq ... \preceq (x_{k+n+1}, y_{k+n+1}) \iff x_k \leq x_{k+1} \leq ... \leq x_{k+n+1} \text{ and } y_k \leq y_{k+1} \leq ... \leq y_{k+n+1}$$

Suppose this chain does not exists. Then, we would have an anti chain of length $n+1$:

$$(x_l, y_l) \not\preceq (x_{l+1}, y_{l+1}) \not\preceq ... \not\preceq (x_{l+n+1}, y_{l+n+1}) \iff \text{Three possibilities:}$$

(i) $x_1 > x_2$ and $y_1 \leq y_2$, in which case we have:

$$x_{l+n+1} \leq x_{l+n} \leq ... \leq x_l \text{ and } y_{l+n+1} \geq y_{l+n} \geq ... \geq y_l$$

(ii) $x_1 \leq x_2$ and $y_1 > y_2$, in which case we have:

$$x_l \leq x_{l+1} \leq ... \leq x_{l+n+1} \text{ and } y_l \geq y_{l+1} \geq ... \geq y_{l+n+1} \geq y_l$$

(iii) $x_1 > x_2$ and $y_1 > y_2$, in which case we have:

$x_{l+n+1} \leq x_{l+n} \leq ... \leq x_l$ and $y_{l+n+1} \leq y_{l+n} \leq ... \leq y_l$ But this is a $n+1$-chain, contradicting our assumption.

Hence, case (iii) does not exists. In any of the other cases the results follow.

(4.20) Divide the number of subsets that contribute at least one monochromatic pair by the number of sets that contain every such pair to get a lower bound on the number of monochromatic pairs:

$$\begin{aligned}
\text{\#of monochromatic pairs} \quad &\geq \quad \frac{\binom{n}{r+1}}{\binom{n-2}{r-1}} \\
&= \quad \frac{\dfrac{n!}{(n-r-1)!(r+1)!}}{\dfrac{(n-2)!}{(n-r-1)!(r-1)!}} \\
&= \quad \frac{n(n-1)}{(r+1)r} \\
&= \quad \frac{n^2-n}{(r+1)r}
\end{aligned}$$

Hence, for a given $r \geq 2$, the number of monochromatic pairs is at least $c \cdot \mathcal{O}(n^2)$, where $c(r) = \frac{1}{(r+1)r}$