Midterm Exam 1

Math-M311 Spring 2011

February 10, 2011

Answer the questions in the spaces provided on the question sheets, being sure to provide full justification for your solutions. If you run out of room for an answer, continue on the back of the page. Your exam should have 6 pages. Please check to make sure your exam is complete.

Name: Solutions

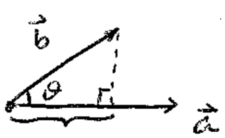
1. (20 points) Let $\vec{a} = \langle 2, 4, 1\rangle$ and $\vec{b} = \langle 1, 1, -1\rangle$ be two vectors in $\mathbb{R}^3$.

   (a) Find $3\vec{a} - \vec{b}$ and compute its norm.

$$3\vec{a} - \vec{b} = \langle 5, 11, 4\rangle$$

$$|3\vec{a} - \vec{b}| = \sqrt{5^2 + 11^2 + 4^2} = \sqrt{25 + 121 + 16} = \sqrt{162} = 9\sqrt{2}.$$

   (b) Compute $\vec{a} \cdot \vec{b}$, and find the vector projection $\text{proj}_{\vec{a}}(\vec{b})$ of $\vec{b}$ onto $\vec{a}$.



$|b|\cos\theta$ $=$ $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

$$\text{proj}_{\vec{a}}(\vec{b}) = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right) \frac{\vec{a}}{\vec{a}} = \frac{5}{21}\langle 2, 4, 1\rangle = \left\langle \frac{10}{21}, \frac{20}{21}, \frac{5}{21} \right\rangle$$

(c) Calculate $\vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(-4-1) - \hat{j}(-2-1) + \hat{k}(2-4)$$

$$= -5\hat{i} + 3\hat{j} - 2\hat{k}$$

(d) Find the area of the triangle which has two of its sides given by $\vec{a}$ and $\vec{b}$.

$$\text{Area} = \frac{1}{2}|\vec{a} \times \vec{b}|$$

$$= \frac{1}{2}\sqrt{25 + 9 + 4}$$

$$= \frac{1}{2}\sqrt{38}$$

2. (20 points) Consider the planes $x + y + z = 1$ and $x + y - z = 1$.

(a) Show that the planes are neither parallel nor perpendicular.

$$\vec{n}_1 = \langle 1, 1, 1 \rangle, \quad \vec{n}_2 = \langle 1, 1, -1 \rangle$$

Not parallel since $\vec{n}_1 \neq \alpha \vec{n}_2$ for some $\alpha \in \mathbb{R}$
(i.e. normals are not parallel)

Not perp. since
$$\vec{n}_1 \cdot \vec{n}_2 = 1 + 1 - 1 = 1 \neq 0.$$

(b) Determine an exact expression for the angle of intersection between these two planes.

$$1 = \vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1| |\vec{n}_2| \cos\theta$$
$$= 3 \cos\theta$$
$$\Longrightarrow \theta = \cos^{-1}\left(\tfrac{1}{3}\right).$$

(c) Find an equation for the line of intersection of these two planes.

Pt. on line is $(1, 0, 0)$.

For dir. vec., take

$$\vec{n} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \hat{\imath}(-1-1) - \hat{\jmath}(-1-1) + \hat{k}(1-1)$$
$$= -2(\hat{\imath} - \hat{\jmath}).$$

$\therefore$ Eqn. of line is
$$\vec{r}(t) = \langle 1, 0, 0 \rangle + t \langle -2, 2, 0 \rangle.$$

3. (a) (20 points) Find the equation of the plane containing the point $(2, 1, 0)$ and that is parallel to the plane $x + 4y - 3z = 1$.

$$(x-2) + 4(y-1) - 3z = 0,$$

i.e. $$x + 4y - 3z = 6.$$

(b) Find the equation of the plane which contains the line with parametric equations $x = 1 + t$, $y = t$, and $z = 3 - 2t$ which is perpendicular to the plane $x - 2y + z = 3$.

For pt., use $(1, 0, 3)$. For normal, use

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix} = -3\langle 1, 1, 1\rangle$$

$\therefore$ Eqn. of plane is

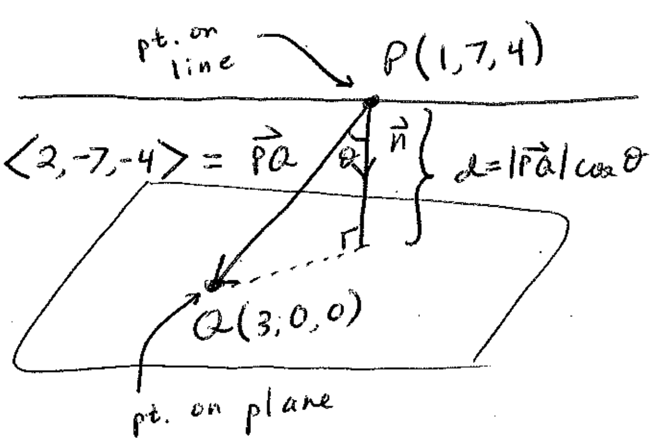
$$(x-1) + y + (z-3) = 0,$$

i.e. $$x + y + z = 4.$$

(c) Consider the plane $3x - y + 4z = 9$ and the line with vector equation $\vec{r}(t) = \langle 1 - 2t, 7 + 6t, 4 + 3t\rangle$. Show the line is parallel to the plane, and find the distance between them.

Line is parallel to plane since

$$\underbrace{\langle 3, -1, 4\rangle}_{\text{normal for plane}} \cdot \underbrace{\langle -2, 6, 3\rangle}_{\text{dir. vec. for line}} = 0.$$

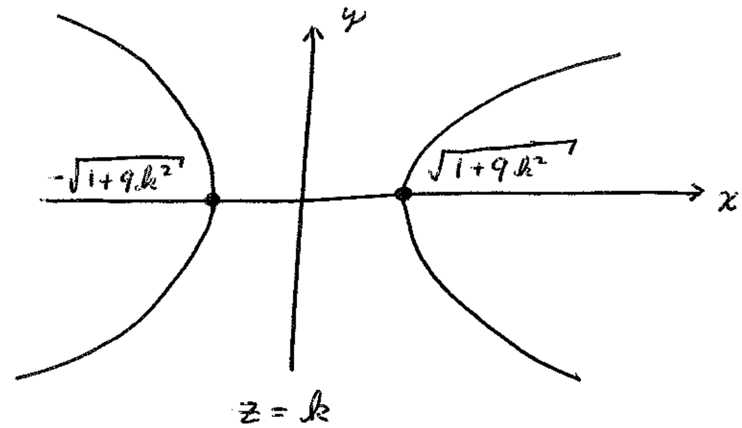


$$d = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|} = \frac{|6 + 7 - 16|}{\sqrt{9 + 1 + 16}} = \frac{3}{\sqrt{26}}$$

4. (20 points) Consider the surface with equation $x^2 - \frac{y^2}{4} - 9z^2 = 1$.

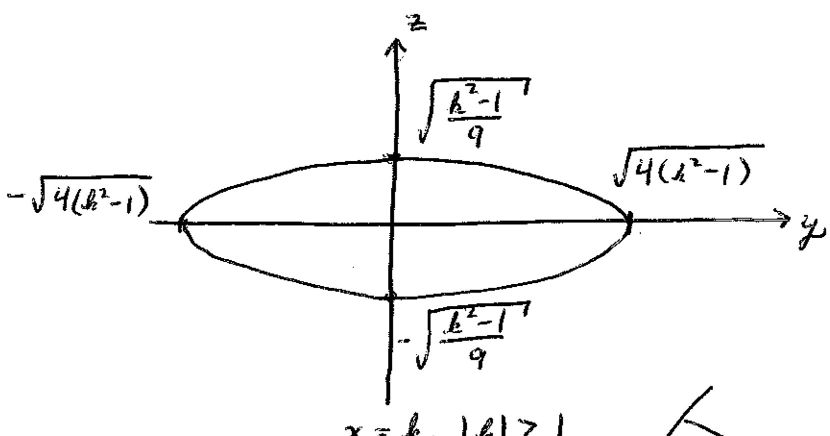
(a) Carefully describe the traces in the planes $z = k$.

$$x^2 - \frac{y^2}{4} = 1 + 9k^2 \Longrightarrow \text{Traces are hyperbolas}$$

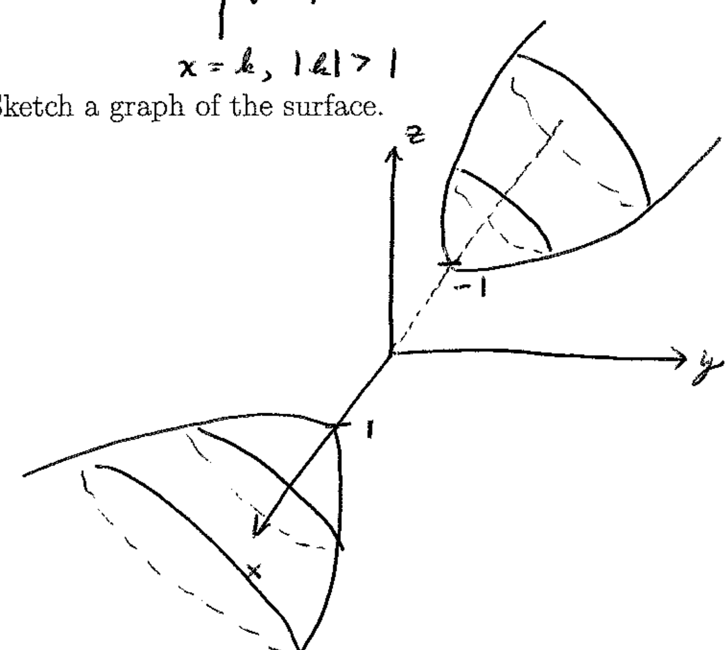


(b) Carefully describe the traces in the planes $x = k$.

$$\frac{y^2}{4} + 9z^2 = k^2 - 1 \Longrightarrow \begin{cases} -1 < k < 1, & \text{No solution!} \\ |k| > 1, & \text{ellipse.} \\ |k| = 1, & (y,z) = (0,0) \end{cases}$$



(c) Sketch a graph of the surface.



Hyperboloid of 2 sheets.

5. (20 points) Suppose a particle is moving along a curve with vector equation $\vec{r}(t)$ such that the the velocity at time t is given by $\vec{r}\,'(t) = \langle 2t^2, \sin(t), \cos(t)\rangle$. Furthermore, suppose the particle passes through the point $(0, -1, 0)$ at $t = 0$.

   (a) From the above information, find the position vector $\vec{r}(t)$.

$$\vec{r}(t) = \langle 0,-1,0\rangle + \int_0^t \langle 2s^2, \sin(s), \cos(s)\rangle \, ds$$

$$= \langle 0,-1,0\rangle + \left\langle \frac{2}{3}s^3, -\cos(s), \sin(s)\right\rangle \Big|_{s=0}^{t}$$

$$= \langle 0,-1,0\rangle + \left\langle \frac{2}{3}t^3, 1-\cos(t), \sin(t)\right\rangle$$

$$= \left\langle \frac{2}{3}t^3, -\cos(t), \sin(t)\right\rangle$$

   (b) Find the equation of the tangent line to the curve $\vec{r}(t)$ at the point $\left(\frac{2}{3}\pi^3, 1, 0\right)$.

First, notice $\vec{r}(\pi) = \langle \frac{2}{3}\pi^3, 1, 0\rangle$ and $\vec{r}\,'(\pi) = \langle 2\pi^2, 0, -1\rangle$. So, eqn. of line is

$$\ell(t) = \left\langle \frac{2}{3}\pi^3, 1, 0\right\rangle + t\langle 2\pi^2, 0, -1\rangle.$$