Midterm Exam 3

Math-M311 Spring 2011

April 21, 2011

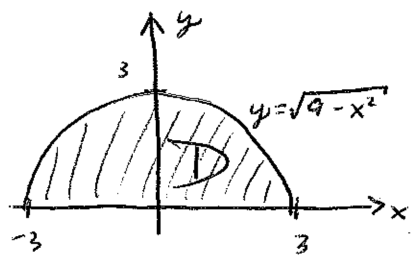
> Answer the questions in the spaces provided on the question sheets, being sure to provide full justification for your solutions. If you run out of room for an answer, continue on the back of the page. Your exam should have 5 pages. Please check to make sure your exam is complete.

Name: Solutions

1. (25 points)

   (a) What is the value of $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} 1\ dy\ dx$?

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} 1\, dy\, dx = \iint_D 1\, dA$$



$$= \text{Area}(D)$$

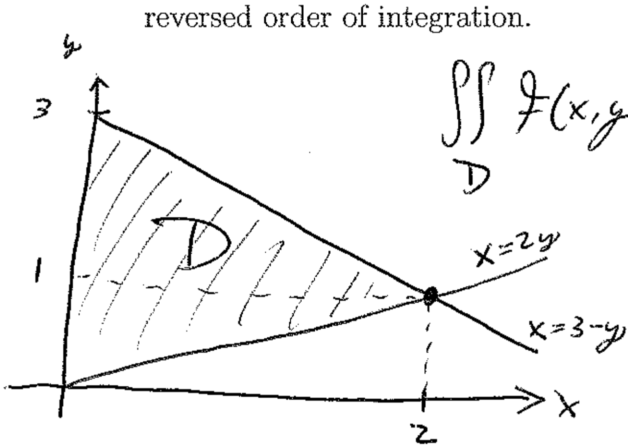
$$= \frac{1}{2}(\pi \cdot 3^2)$$

$$= \frac{9\pi}{2}.$$

   (b) In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

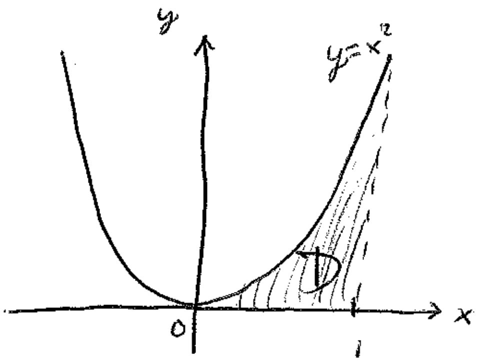
$$\int\int_D f(x,y)dA = \int_0^1\int_0^{2y} f(x,y)dx\ dy + \int_1^3\int_0^{3-y} f(x,y)dx\ dy.$$

Sketch the region D and express the double integral as an iterated integral with reversed order of integration.



$$\iint_D f(x,y)\,dA = \int_0^2\int_{\frac{x}{2}}^{3-x} f(x,y)\,dy\,dx.$$

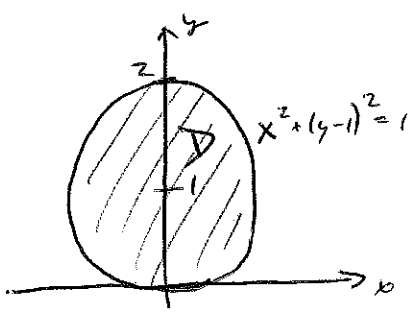
(c) Evaluate $\int\int_D \sqrt{x^3+1}\, dA$ where D is the region in the xy-plane bounded by the curves $y=0$, $y=x^2$, $x=0$, and $x=1$.



$$\iint_D \sqrt{x^3+1}\, dA = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx$$
$$= \int_0^1 x^2 \sqrt{x^3+1}\, dx$$
$$= \frac{2}{9}(x^3+1)^{3/2} \Big|_{x=0}^{1}$$
$$= \frac{2}{9}\left(2^{3/2}-1\right).$$

(d) Set up a double integral in polar coordinates to find the area of the region bounded by the curve $x^2+y^2-2y=0$. (*Note: You will receive no credit for solving this problem without polar coordinates...*)

$$x^2+y^2-2y=0 \implies x^2+(y-1)^2=1$$



In polar coords., $x=r\cos\theta$, $y=r\sin\theta$,

$x^2+y^2-2y=0$ becomes

$$r^2-2r\sin\theta=0$$
$$\implies r=2\sin\theta,\ \theta\in[0,\pi].$$

So, $$\text{Area}(D)=\iint_D 1\, dA$$
$$=\int_0^{\pi}\int_0^{2\sin\theta} r\, dr\, d\theta.$$

2. (25 points) Consider the function $f(x, y) = x^2 + 3y^2$.

(a) Use the method of Lagrange multipliers to find the extreme values of f on the ellipse $\underbrace{2x^2 + y^2}_{g(x,y)} = 3$.

Need to solve

$$\begin{cases} 2x = 4\lambda x & (1) \\ 6y = 2\lambda y & (2) \\ 2x^2 + y^2 = 3 & (3) \end{cases}$$

$(1) \Rightarrow x = 0$ or $\lambda = \frac{1}{2}$.

If $x = 0$, $(3) \Rightarrow y = \pm\sqrt{3}$ so $(0, \pm\sqrt{3})$ solve (1)-(3) for some λ.

If $\lambda = \frac{1}{2}$, $(2) \Rightarrow 6y = y \Rightarrow y = 0$ and $(3) \Rightarrow x = \pm\sqrt{\frac{3}{2}}$, so $(\pm\sqrt{\frac{3}{2}}, 0)$ solve (1)-(3).

Extreme values of f on $g(x,y) = 3$ are

$$\begin{cases} f(0, \pm\sqrt{3}) = 9 \\ f(\pm\sqrt{\frac{3}{2}}, 0) = \frac{3}{2} \end{cases}$$

(b) Find the absolute max and min values of f on the region $2x^2 + y^2 \leq 3$.

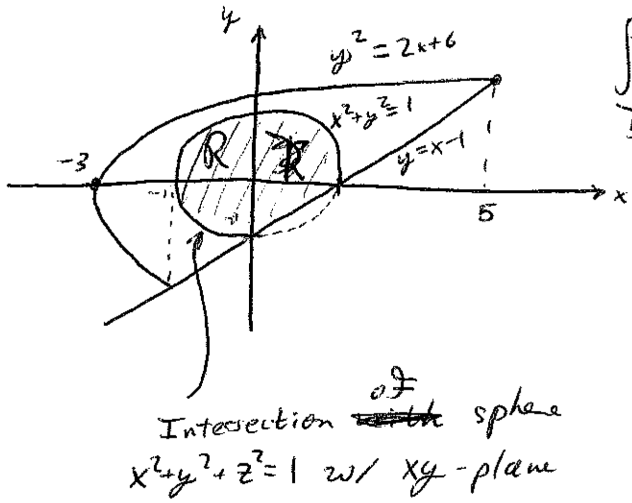
Notice $\nabla f = 0 \Rightarrow 2x = 0$ and $6y = 0$. Thus $(0,0)$ is only critical pt. of f, and $f(0,0) = 0$.

Abs Max $= 9 = f(0, \pm\sqrt{3})$

Abs Min $= 0 = f(0,0)$.

3. (25 points) Set up, but do not evaluate, the following integrals as iterated integrals.

(a) $\int\int\int_D f(x,y,z)dV$ where D is the solid region region bounded by the sphere $x^2+y^2+z^2=1$ and the surfaces $y=x-1$, $y^2=2x+6$.

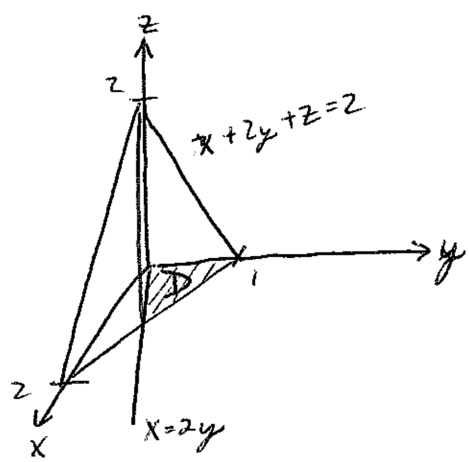


$$\iiint_D f(x,y,z)\,dV = \iint_R \left[\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\right] dA_{xy}$$

$$= \int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left[\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\right] dy\,dx$$

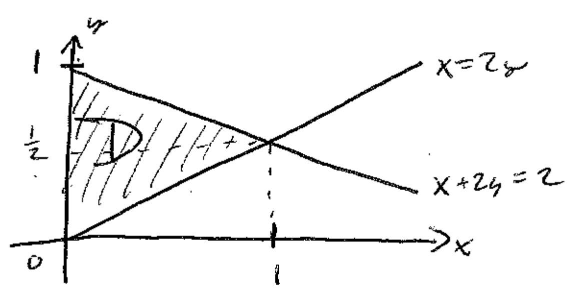
$$+ \int_{0}^{1}\int_{x-1}^{\sqrt{1-x^2}}\left[\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\right] dy\,dx.$$

(b) $\int\int\int_E f(x,y,z)dV$ where E is the tetrahedron bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$, and $z=0$.



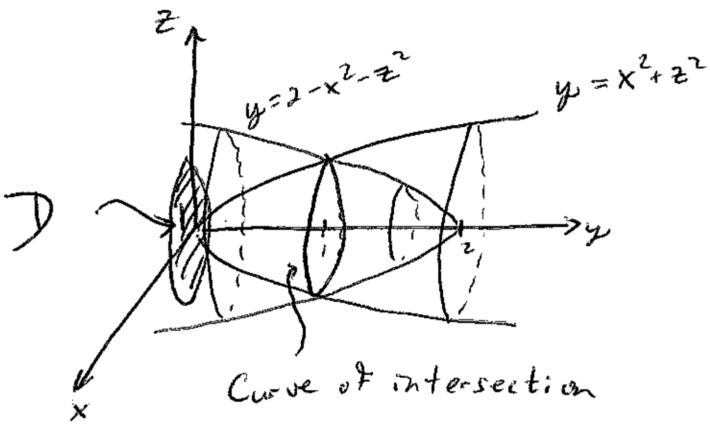
$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_0^{2-x-2y} f(x,y,z)\,dz\right] dA_{xy}$$

$$= \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} \left[\int_0^{2-x-2y} f(x,y,z)\,dz\right] dy\,dx.$$



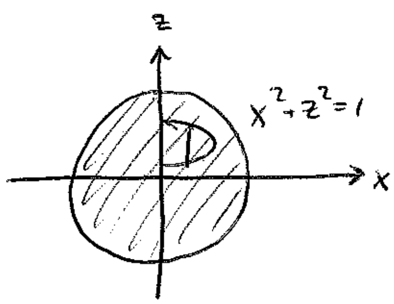
dV... oops again!

4. (25 points) Evaluate the triple integral $\int\int\int_E y\sqrt{x^2+z^2}$ where E is the solid region bounded by the paraboloids $y = x^2+z^2$ and $y = 2-x^2-z^2$.



is $x^2+z^2 = 2-x^2-z^2$, $y=1$

$\Longrightarrow x^2+z^2=1$, $y=1$.



Use polar coords. in xz-plane:

$\begin{cases} x = r\cos\theta \\ z = r\sin\theta \end{cases}$

$$\iiint_E y\sqrt{x^2+z^2}\,dV$$

$$= \iint_D \left[\int_{x^2+z^2}^{2-x^2-z^2} y\sqrt{x^2+z^2}\,dy\right] dA_{xz}$$

$$= \int_0^{2\pi}\int_0^1 \left[\int_{r^2}^{2-r^2} y r\,dy\right] r\,dr\,d\theta$$

$$= 2\pi\int_0^1 \left[\frac{r^2}{2}\, y^2\Big|_{y=r^2}^{2-r^2}\right] dr$$

$$= \pi\int_0^1 r^2\left[(2-r^2)^2 - r^4\right] dr$$

$$= \pi\int_0^1 r^2(4-4r^2)\,dr$$

$$= 4\pi\left(\frac{1}{3}r^3 - \frac{1}{5}r^5\Big|_{r=0}^{1}\right)$$

$$= 4\pi\left(\frac{1}{3}-\frac{1}{5}\right)$$

$$= \frac{8\pi}{15}.$$