1. Let $\vec{F} = 2yz\,i + (-x+3y+2)\,j + (x^2+z)\,k$. Evaluate

$$\iint_S (\nabla \times \vec{F}) \cdot dS,$$

where $S$ is the cylinder $x^2+y^2=a^2$, $0 \leq z \leq 1$ (without the top and bottom). What if the top and bottom are included?

<u>Solution</u> Let $R$ be the surface $S \cup D$, where $D = \{(x,y,z) : x^2+y^2 \leq a^2, z=1\}$.

Notice that $\iint_S (\nabla\times\vec{F})\cdot dS = \iint_R (\nabla\times\vec{F})\cdot dS - \iint_D (\nabla\times\vec{F})\cdot dS$.

We also let $D' = \{(x,y,z) : x^2+y^2 \leq a^2, z=0\}$.

By Stokes' theorem,

$$\begin{aligned}\iint_R (\nabla\times\vec{F})\cdot dS &= \int_{\partial D'} 2yz\,dx + (-x+3y+2)\,dy \qquad (dz=0,\ \partial R = \partial D')\\ &= \int_{\partial D'} (-x+3y+2)\,dy \qquad (z=0)\\ &= \int_0^{2\pi} (-a\cos t + 3a\sin t + 2)(a\cos t)\,dt\\ &= -a^2\pi\end{aligned}$$

On the other hand, by Stokes' theorem again,

$$\begin{aligned}\iint_D (\nabla\times\vec{F})\cdot dS &= \int_0^{2\pi} (2a\sin t)(-a\sin t) + (-a\cos t + 3a\sin t + 2)(a\cos t)\,dt\\ &= \int_0^{2\pi} -2a^2\sin^2 t - a^2\cos^2 t\,dt\\ &= -3a^2\pi.\end{aligned}$$

Hence $\iint_S (\nabla\times\vec{F})\cdot dS = -a^2\pi - (-3a^2\pi) = 2\pi a^2$.

If top and bottom are included, then the surface has no boundary, and Stokes' theorem tells us that the integral is 0.

3. Let $\vec{F} = x^2y\,i + z^8\,j - 2xyz\,k$. Evaluate the integral of $\vec{F}$ over the surface of the unit cube.

<u>Solution</u> Note that $\nabla\cdot\vec{F} = 2xy + 0 - 2xy = 0$.

Gauss' theorem $\Rightarrow \iint_{\partial \text{cube}} \vec{F}\cdot n\,dS = \iiint_{\text{cube}} \nabla\cdot\vec{F}\,dS = 0$.

4. Verify Green's theorem for the line integral

$$\int_C x^2y\,dx + y\,dy,$$

when $C$ is the boundary of the region between the curves $y=x$ and $y=x^3$, $0 \leq x \leq 1$.

Solution We first find the line integral by definition.

Parametrize the two segment of the curves by

$\gamma_1(t) = (t, t^3)$, $t \in [0, 1]$,

$\gamma_2(t) = (t, t)$, $t$ from 1 to 0.

Then
$$\int_C x^2 y\, dx + y\, dy = \int_0^1 t^2(t^3) + t^3(3t^2)\, dt + \int_1^0 t^3 + t\, dt$$
$$= \int_0^1 4t^5 - t^3 - t\, dt$$
$$= \left[\frac{4t^6}{6} - \frac{t^4}{4} - \frac{t^2}{2}\right]_0^1$$
$$= \frac{2}{3} - \frac{1}{4} - \frac{1}{2} = \frac{-1}{12}.$$

Now we verify the Green's theorem.
$$\int_C x^2 y\, dx + y\, dy = \int_0^1 \int_{x^3}^{x} (0 - x^2)\, dy\, dx$$
$$= \int_0^1 (x^5 - x^3)\, dx$$
$$= \frac{1}{6} - \frac{1}{4} = \frac{-1}{12}.$$

Hence both integrals are equal and Green's theorem is verified in this case.

5(a) Show that $\vec{F} = (x^3 - 2xy^3)i - 3x^2y^2 j$ is a gradient vector field.

Solution Consider $f(x,y) = \frac{x^4}{4} - x^2y^3$. Check $\nabla f = \vec{F}$.

(b) Evaluate the integral of $\vec{F}$ along the path $x = \cos^3\theta$, $y = \sin^3\theta$, $0 \le \theta \le \frac{\pi}{2}$.

Solution By (a), $\vec{F}$ is a gradient vector field, and the integral is independent of path.

Thus $\int \vec{F} \cdot ds = f(0, 1) - f(1, 0) = 0 - \frac{1}{4} = -\frac{1}{4}$.

7(a) Show that $\vec{F} = 6xy(\cos z)i + 3x^2(\cos z)j - 3x^2y(\sin z)k$ is conservative.

Solution
$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy(\cos z) & 3x^2(\cos z) & -3x^2y(\sin z) \end{vmatrix}$$
$$= (-3x^2 \sin z + 3x^2 \sin z)i - (-6xy \sin z + 6xy \sin z)j + (6x \cos z - 6x \cos z)k$$
$$= 0.$$

$\therefore \vec{F}$ is conservative.

(b) Find $f$ such that $\vec{F} = \nabla f$.

Solution Let $f(x,y,z) = 3x^2 y \cos z$. Check $\nabla f = \vec{F}$.

(c) Evaluate the integral of $\vec{F}$ along the curve $x = \cos^3\theta$, $y = \sin^3\theta$, $z=0$, $0 \le \theta \le \frac{\pi}{2}$.

Solution $\int \vec{F} \cdot ds = f(0,1,0) - f(1,0,0) = 0 - 0 = 0$.

11. Let $\vec{a}$ be a constant vector and $\vec{F} = \vec{a} \times \vec{r}$ $(\vec{r}(x,y,z) = (x,y,z))$. Is $\vec{F}$ conservative? If so, find a potential for it.

Solution Write $\vec{a} = (a_1, a_2, a_3)$.

Then

$$\vec{a} \times \vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= (a_2 z - a_3 y)i - (a_1 z - a_3 x)j + (a_1 y - a_2 x)k.$$

$\therefore$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= 2a_1 i + 2a_2 j + 2a_3 k.$$

$\therefore \nabla \times \vec{F} = 0$ only when $\vec{a} = 0$.

$\therefore \vec{F}$ is conservative if and only if $\vec{a} = 0$.

If $\vec{a} = 0$, then $\vec{F} = 0$, and so $0$ is a potential for $\vec{F}$.

12. Show that the fields $\vec{F}$ in (a) and (b) are conservative and find a function $f$ such that $\vec{F} = \nabla f$.

(a) $\vec{F} = y^2 e^{xy^2} i + 2y e^{xy^2} j$

(b) $\vec{F} = (\sin y)i + (x \cos y)i + (e^z)k$.

Solution (a) I believe that the problem is wrong. $\vec{F}$ is not conservative at all.

Instead let's consider $(y^2 e^{xy^2})i + (2xy e^{xy^2})j$.

In this case, $\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y^2 e^{xy^2} & 2xy e^{xy^2} \end{vmatrix} = \begin{array}{l} 2y e^{xy^2} + 4xy^2 e^{xy^2} \\ - 2y e^{xy^2} - 4xy^2 e^{xy^2} \end{array} = 0.$

$\therefore \vec{F}$ is conservative.

A possible choice of $f$ is $f(x,y) = e^{xy^2}$.

(b)
$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & x\cos y & e^z \end{vmatrix}$$
$$= 0i + 0j + (\cos y - \cos y)k = 0.$$

$\therefore \vec{F}$ is conservative.

A possible choice of $f$ is $x\sin y + e^z$.

13(a) Let $f(x,y,z) = 3xye^{z^2}$. Compute $\nabla f$.

<u>Solution</u> $\nabla f = (3ye^{z^2},\ 3xe^{z^2},\ 6xyze^{z^2})$.

(b) Let $\vec{c}(t) = (3\cos^3 t, \sin^2 t, e^t)$, $0 \le t \le \pi$. Evaluate $\int_{\vec{c}} \nabla f \cdot ds$.

<u>Solution</u> By independence of path,
$$\int_{\vec{c}} \nabla f \cdot ds = f(-3, 0, e^\pi) - f(3, 0, 1) = 0 - 0 = 0.$$

(c) Verify directly Stokes' theorem for gradient vector fields $\vec{F} = \nabla f$.

<u>Solution</u> We just need to check whether $\int \nabla \times \vec{F} \cdot dS = 0$.
$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3ye^{z^2} & 3xe^{z^2} & 6xyze^{z^2} \end{vmatrix}$$
$$= (6xze^{z^2} - 6xze^{z^2})i - (6yze^{z^2} - 6yze^{z^2})j + (3e^{z^2} - 3e^{z^2})k$$
$$= 0.$$

$\therefore \iint \nabla \times \vec{F} \cdot dS = 0$.

14. Using Green's theorem, or otherwise, evaluate $\int_C x^3 dy - y^3 dx$, where $C$ is the unit circle $(x^2 + y^2 = 1)$.

<u>Solution</u> Let $D = \{(x,y) : x^2 + y^2 \le 1\}$.

By Green's theorem,

$$\int_C -y^3\,dx + x^3\,dy = \iint_D 3x^2+3y^2\,dx\,dy$$
$$= \int_0^{2\pi}\int_0^1 3r^2\cdot r\,dr\,d\theta \quad \text{(polar coordinates)}$$
$$= \frac{3}{2}\pi$$

Remark: We can also compute it directly.

$$\int_C x^3dy - y^3dx = \int_0^{2\pi} (\cos^3 t)(\cos t) - (\sin^3 t)(-\sin t)\,dt$$
$$= \int_0^{2\pi} \sin^4 t + \cos^4 t\,dt$$
$$= \frac{3}{2}\pi \quad \text{(Verify this !)}$$

15. Evaluate the integral $\iint_S F\cdot dS$, where $\vec{F} = xi + yj + 3k$ and where $S$ is the surface of unit sphere $x^2+y^2+z^2=1$.

Solution Note that $\nabla\cdot\vec{F} = 2$.

$\therefore$ Gauss' theorem $\Rightarrow \iint_S F\cdot dS = \iiint_{x^2+y^2+z^2\le 1} 2\,dV = \frac{8}{3}\pi$.

(Recall the volume of a ball of radius $r$ is $\frac{4}{3}\pi r^3$).

17. Use Green's theorem to find the area of the loop of the curve $x = a\sin\theta\cos\theta$, $y = a\sin^2\theta$, for $a>0$ and $0\le\theta\le\pi$.

Solution Green's theorem $\Rightarrow$ Area $= \int x\,dy$

$$= \int_0^{\pi} a\sin\theta\cos\theta\cdot 2a\sin\theta\cos\theta\,d\theta$$
$$= 2a^2\int_0^{\pi}\sin^2\theta\cos^2\theta\,d\theta$$
$$= \frac{a^2}{2}\int_0^{\pi}\sin^2 2\theta\,d\theta$$
$$= \frac{\pi}{4}a^2.$$

(if you forget how to integrate, recall $\sin^2 2\theta = \frac{1}{2} - \frac{1}{2}\cos(4\theta)$)

18. Evaluate $\int_C yz\,dx + xz\,dy + xy\,dz$, where $C$ is the curve of intersection of the cylinder $x^2+y^2=1$ and the surface $z=y^2$.

Solution We evaluate it via Stokes' theorem. Let $\vec{F} = yzi + xzj + xyk$.

Now,

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$
$$= 0.$$

$$\therefore \int_C \vec{F}\cdot ds = \iint \nabla\times\vec{F}\cdot dS = 0.$$

19. Evaluate $\int_C (x+y)\,dx + (2x-z)\,dy + (y-z)\,dz$, where $C$ is the perimeter of the triangle connecting $(2,0,0)$, $(0,3,0)$, $(0,0,6)$, in that order.

Solution Again we will use Stokes' theorem.

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y-z \end{vmatrix}$$

$$= 2i + k.$$

Notice that the plane $3x+2y+z=6$ contains the triangle.

$$\therefore (\nabla\times\vec{F})\cdot\vec{n} = (2,0,1)\cdot\frac{1}{\sqrt{14}}(3,2,1) = \frac{1}{\sqrt{14}}(6+1) = \sqrt{\frac{7}{2}}$$

Hence $\int_C (x+y)\,dx + (2x-z)\,dy + (y-z)\,dz$

$$= \iint_\Delta \nabla\times\vec{F}\cdot d\vec{S}$$

$$= \sqrt{\frac{7}{2}}\ \text{Area}(\Delta)$$

$$\text{Area of } \Delta = \left\| \frac{1}{2} \begin{vmatrix} i & j & k \\ 2 & 0 & -6 \\ 0 & 3 & -6 \end{vmatrix} \right\|$$

($\|\cdot\|$ means length of a vector here)

$$= \left\| \frac{1}{2}(18, 12, 6) \right\|$$

$$= \|(9, 6, 3)\|$$

$$= \sqrt{126}$$

$$\therefore \sqrt{\frac{7}{2}}\ \text{Area}(\Delta) = \sqrt{\frac{7}{2}}\cdot\sqrt{126} = \sqrt{\frac{7}{2}}\sqrt{7\cdot 18} = 7\cdot 3 = 21.$$

20. Which of the following are conservative fields on $\mathbb{R}^3$? For those that are, find a function $f$ such that $\vec{F} = \nabla f$.

(a) $\vec{F}(x,y,z) = 3x^2y\,i + x^3\,j + 5k$.

(b) $\vec{F}(x,y,z) = (x+z)\,i - (y+z)\,j + (x-y)\,k$.

(c) $\vec{F}(x,y,z) = 2xy^3 + x^2z^3\,j + 3x^2yz^2\,k$.

Solution (a)

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y & x^3 & 5 \end{vmatrix}$$

$$= 0i + 0j + (3x^2 - 3x^2)k = 0.$$

$\therefore \vec{F}$ is conservative.

$f(x,y,z) = x^3y + 5z$ is a possible choice.

(b)
$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+z & -y-z & x-y \end{vmatrix}$$
$$= 0i + 0j + 0k$$

$\therefore \vec{F}$ is conservative.

Check $f(x,y,z) = \frac{x^2}{2} + xz - \frac{y^2}{2} - yz$ works.

(c).
$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix} = (3x^2z^2 - 3x^2z^2)i - (6xyz^2 - 0)j + (2xz^3 - 6xy^2)k$$
$$= -6xyz^2 j + (2xz^3 - 6xy^2)k \neq 0.$$

$\therefore \vec{F}$ is not conservative.

21. Consider the following two vector fields in $\mathbb{R}^3$:

(i) $\vec{F}(x,y,z) = y^2 i - z^2 j + x^2 k$.

(ii) $\vec{G}(x,y,z) = (x^3 - 3xy^2)i + (y^3 - 3x^2y)j + zk$.

(a) Which of these fields (if any) are conservative on $\mathbb{R}^3$? Give reasons for your answer.

Solution
$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -z^2 & x^2 \end{vmatrix} = 2zi - 2xj - 2yk \neq 0$$

$\therefore \vec{F}$ is not conservative.
$$\nabla\times\vec{G} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3-3xy^2 & y^3-3x^2y & z \end{vmatrix}$$
$$= 0i + 0j + (-6xy + 6xy)k = 0$$

$\therefore \vec{G}$ is conservative.

(b) Find potential for fields that are conservative.

Solution This time I do it step by step.
$$\int (x^3 - 3xy^2)\,dx = \underbrace{\frac{x^4}{4} - \frac{3}{2}x^2y^2 + g(y,z)}_{\text{This will be our potential, call it } f} \text{ for some function } g.$$
$$\frac{\partial f}{\partial y} = y^3 - 3x^2y \Rightarrow -3x^2y + \frac{\partial g}{\partial y} = y^3 - 3x^2y \Rightarrow \frac{\partial g}{\partial y} = y^3$$

$\therefore\ g(y,z) = \frac{y^4}{4} + h(z)$ for some function $h$.

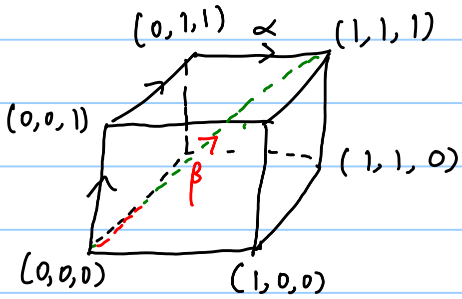
$\frac{\partial f}{\partial z} = z \Rightarrow h'(z) = z \Rightarrow h(z) = \frac{z^2}{2} + c$ for some constant $c$.

$\therefore\ f(x,y,z) = \frac{x^4}{4} - \frac{3}{2}x^2y^2 + \frac{y^4}{4} + \frac{z^2}{2} + c$ is a potential for $\vec{G}$.

(c) Let $\alpha$ be the path that goes from $(0,0,0)$ to $(1,1,1)$ by following edges of the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ from $(0,0,0)$ to $(0,0,1)$ to $(0,1,1)$ to $(1,1,1)$.

Let $\beta$ be the path from $(0,0,0)$ to $(1,1,1)$ directly along the diagonal of the cube.

Find $\int_\alpha \vec{F}\cdot ds$, $\int_\alpha \vec{G}\, ds$, $\int_\beta \vec{F}\cdot ds$, $\int_\beta \vec{G}\cdot ds$




Solution $\int_\alpha \vec{F}\cdot ds$: Parametrize $\alpha$ by a series of curves:

$$\begin{cases} \gamma_1(t) = (0,0,t), & t \in [0,1] \\ \gamma_2(t) = (0,t,1), & t \in [0,1] \\ \gamma_3(t) = (t,1,1), & t \in [0,1] \end{cases}$$

Then
$$\int_\alpha \vec{F}\cdot ds = \int_0^1 0+0+0\,dt + \int_0^1 0 + (-1^2)\cdot(1) + 0\,dt + \int_0^1 1\cdot 1 + 0 + 0\,dt = 1 - 1 = 0.$$

$\int_\beta \vec{F}\cdot ds$: Parametrize $\beta$ by $\gamma(t) = (t,t,t)$, $t \in [0,1]$.

Then
$$\int_\beta \vec{F}\cdot ds = \int_0^1 t^2\cdot 1 - t^2\cdot 1 + t^2\cdot 1\,dt = \int_0^1 t^2\,dt = \frac{1}{3}.$$

$\int_\alpha \vec{G}\cdot ds$, $\int_\beta \vec{G}\cdot ds$: Two integrals are the same as $\vec{G}$ is conservative.

They both equal $f(1,1,1) - f(0,0,0) = \frac{1}{4} - \frac{3}{2} + \frac{1}{4} + \frac{1}{2} + c - c = -\frac{1}{2}$.

Good luck to your final exam! ☺