M312 - HW 10 - Fall 2013 - Enrique Areyan

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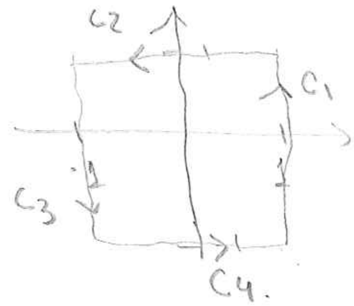
SECTION 8.1:

(4) Verify Green's theorem:

$D=[-1,1]\times[-1,1]$, $P(x,y)=x$, $Q(x,y)=y$



Solution: we want to verify that

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Let us compute each side separately:

$$\int_{C^+} P\,dx+Q\,dy = \int_{C_1} P\,dx+Q\,dy + \int_{C_2} P\,dx+Q\,dy + \int_{C_3} P\,dx+Q\,dy + \int_{C_4} P\,dx+Q\,dy.$$

$C_1(t)=(1,t)$; $-1\le t\le 1$

$$\int_{C_1} P\,dx+Q\,dy = \int_{-1}^{1}\left(P(c_1(t))\cdot \overset{0}{0} + Q(c_1(t))\cdot 1\right)dt = \int_{-1}^{1} t\,dt = \left[\frac{t^2}{2}\right]_{-1}^{1} = \frac{1}{2}-\frac{1}{2} = \boxed{0}$$

$C_2^-(t)=(t,1)$; $-1\le t\le 1$.

$$\int_{C_2} P\,dx+Q\,dy = -\int_{C_2^-} P\,dx+Q\,dy = -\int_{-1}^{1}\left(P(\bar{c}_2(t))1 + Q(\bar{c}_2(t))\cdot \overset{0}{0}\right)dt = -\int_{-1}^{1} t\,dt = \boxed{0}$$

$C_3^-(t)=(-1,t)$; $-1\le t\le 1$.

$$\int_{C_3} P\,dx+Q\,dy = -\int_{C_3^-} P\,dx+Q\,dy = -\int_{-1}^{1}\left(P(\bar{c}_3(t))\cdot \overset{0}{0} + Q(\bar{c}_3(t))\cdot 1\right)dt = -\int_{-1}^{1} t\,dt = \boxed{0}$$

$C_4(t)=(t,-1)$; $-1\le t\le 1$.

$$\int_{C_4} P\,dx+Q\,dy = \int_{-1}^{1}\left(P(c_4(t))\cdot 1 + Q(c_4(t))\cdot \overset{0}{0}\right)dt = \int_{-1}^{1} t\,dt = \boxed{0}$$

$$\int_{C^+} P\,dx+Q\,dy = 0+0+0+0 = \boxed{0}.$$

Now, the right-hand side:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{-1}^{1}\int_{-1}^{1} 0-0\,dx\,dy = \boxed{0}.$$

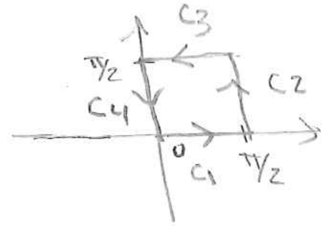
We have verified Green's theorem for this particular example.

(6) Verify Green's theorem.

$D = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$, $P(x,y) = \sin x$, $Q(x,y) = \cos y$.

Solution: We want to verify that:

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



$C_1(t) = (t, 0)$; $0 \le t \le \pi/2$.

$$\int_{C_1} P\,dx + Q\,dy = \int_{C_1} P(C_1(t))\cdot 1 + Q(C_1(t))\cdot 0 = \int_0^{\pi/2} \sin t\,dt = \left[-\cos t\right]_0^{\pi/2} = \left[-\cos(\pi/2) + \cos(0)\right] = \boxed{1}$$

$C_2(t) = (\pi/2, t)$; $0 \le t \le \pi/2$.

$$\int_{C_2} P\,dx + Q\,dy = \int_{C_2} P(C_2(t))\cdot 0 + Q(C_2(t))\cdot 1 = \int_0^{\pi/2} \cos t\,dt = \left[\sin t\right]_0^{\pi/2} = \left[\sin(\pi/2) - \sin(0)\right] = \boxed{1}$$

$C_3^-(t) = (t, \pi/2)$; $0 \le t \le \pi/2$

$$\int_{C_3} P\,dx + Q\,dy = -\int_{C_3^-} P(C_3^-(t))\cdot 1 + Q(C_3^-(t))\cdot 0 = -\int_0^{\pi/2} \sin t\,dt = \left[\cos t\right]_0^{\pi/2} = \left[\cos(\pi/2) - \cos(0)\right] = \boxed{-1}$$

$C_4^-(t) = (0, t)$; $0 \le t \le \pi/2$

$$\int_{C_4} P\,dx + Q\,dy = -\int_{C_4^-} P(C_4^-(t))\cdot 0 + Q(C_4^-(t))\cdot 1 = -\int_0^{\pi/2} \cos t = \left[\sin t\right]_0^{\pi/2} = \left[-\sin \pi/2 + \sin 0\right] = \boxed{-1}$$

Hence, $\int_{C^+} P\,dx + Q\,dy = 1 - 1 + 1 - 1 = \boxed{0}$. Now, the right-hand side.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_0^{\pi/2}\int_0^{\pi/2} (0 - 0)\,dx\,dy = \boxed{0}$$

(9) Evaluate $\int_C y\,dx - x\,dy$, where C is the boundary of the square $[-1,1]\times[-1,1]$ oriented in the counter clockwise direction, using Green's theorem.

Solution: According to Green's theorem:

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \,,$$ in our case:

$P(x,y) = y$ and $Q(x,y) = -x$ $D = [-1,1]\times[-1,1]$. Hence,

$$\int_C y\,dx - x\,dy = \int_{-1}^{1}\int_{-1}^{1} (-1-1)\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} -2\,dx\,dy = -2(2)(2) = \boxed{-8}$$

(11) Verify Green's theorem for the disc D with center (0,0) and radius R and the functions:

(a) $P(x,y) = xy^2$, $Q(x,y) = -yx^2$

For the boundary of the disc use the parametrization: (we will use this param. for all other parts)

$c(t) = (R\cos(t), R\sin(t))$ $0 \le t \le 2\pi$

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$ Let us compute each side separately:

$$\int_{C^+} P\,dx + Q\,dy = \int_0^{2\pi} \left[P(c(t))\,R(-\sin t) + Q(c(t))\,R(\cos t)\right] dt$$

$$= \int_0^{2\pi} (R\cos t)(R^2\sin^2 t)\,R(-\sin t) + (-R\sin(t))(R^2\cos^2 t)\,R\cos t\; dt$$

$$= \int_0^{2\pi} -R^4\cos t\,\sin^3 t - R^4\sin(t)\cos^3 t\; dt$$

$$= -R^4\int_0^{2\pi} \cos t\,\sin^3 t - \sin(t)\cos^3 t\; dt = \boxed{0}$$. Now, the right-hand side:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D -2xy - 2xy\; dx\,dy = -4\iint_{x^2+y^2\le R^2} xy\; dx\,dy$$

$$-4\int_0^{2\pi}\int_0^{R^2} r^3\cos\theta\sin\theta\; dr\,d\theta = -4\int_0^{2\pi}\cos\theta\sin\theta\left[\frac{r^4}{4}\right]_0^{R^2} d\theta = -R^8\int_0^{2\pi}\cos\theta\sin\theta\,d\theta = \frac{+R^8}{2}\left[\cos^2(\theta)\right]_0^{2\pi} = \frac{+R^8}{2}[1-1] = \boxed{0}$$

(b) $P(x,y) = x+y$, $Q(x,y) = y$

$$\int_{C^+} P\,dx + Q\,dy = \int_0^{2\pi} P(c(t))R(-\sin t) + Q(c(t))\,R\cos t\,dt$$

$$= \int_0^{2\pi} R^2(\cos t+\sin t)(-\sin t) + R^2 \sin t \cos t\,dt$$ $\to 0$ (by previous part)

$$= -R^2\int_0^{2\pi} \sin t\cos t + \sin^2 t\,dt = -R^2\Big[\underbrace{\int_0^{2\pi}\sin t\cos t\,dt}_{\to 0} + \int_0^{2\pi}\sin^2 t\,dt\Big]$$ (by previous part

$$= -R^2\int_0^{2\pi}\sin^2 t\,dt = \frac{-R^2}{2}\Big[t-\sin(t)\cos(t)\Big]_0^{2\pi}$$

$$= -\frac{R^2}{2}\Big[(2\pi-\sin(2\pi)\cos(2\pi)) - (0-\sin(0)\cos(0))\Big] = \boxed{-\pi R^2}$$

$$\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy = \iint_{x^2+y^2\le R^2} 0-1\,dx\,dy = \iint_{x^2+y^2\le R^2} -1\,dx\,dy = -\text{Area(circle of radius } R)$$
$$= \boxed{-\pi R^2}$$

(c) $P(x,y) = xy = Q(x,y)$.

$$\int_{C^+} P\,dx+Q\,dy = \int_{C^+} P(c(t))(-R\sin t) + Q(c(t))(R\cos t)$$

$$= \int_{C^+} (R\cos t)(R\sin t)(-R\sin t) + (R\cos(t))(R\sin(t))(R\cos t)\,dt$$

by part (a)

$$= \int_{C^+} R^3\cos^2 t\sin t - R^3\cos t\sin^2 t\,dt = -R^3\int_0^{2\pi}\cos t\sin^2 t - \sin t\cos^2 t\,dt = \boxed{0}$$

$$\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy = \iint_D y-x\,dx\,dy = \int_0^{2\pi}\int_0^R r^2(\sin\theta-\cos\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\sin\theta-\cos\theta\left[\frac{r^3}{3}\right]_0^R d\theta = \frac{R^3}{3}\int_0^{2\pi}\sin\theta-\cos\theta\,d\theta$$

$$= \frac{R^3}{3}\left[\int_0^{2\pi}\sin\theta\,d\theta - \int_0^{2\pi}\cos\theta\,d\theta\right] = \frac{R^3}{3}\Big[[-\cos\theta]_0^{2\pi} - [\sin\theta]_0^{2\pi}\Big]$$

$$= \frac{R^3}{3}\Big[(-\cos 2\pi+\cos(0)) - (\sin 2\pi - \sin 0)\Big] = \frac{R^3}{3}[-1+1] = \frac{R^3}{3}\cdot 0 = \boxed{0}$$

(d) $P(x,y) = 2y$, $Q(x,y) = x$.

$$\int_{C^+} P\,dx + Q\,dy = \int_{C^+} P(c(t))(R(-\sin t)) + Q(c(t))(R\cos t)$$

$$= \int_{C^+} 2R\sin t(R(-\sin t)) + (R\cos t)R\cos t$$

$$= \int_{C^+} -2R^2\sin^2 t + R^2\cos^2 t\,dt = R^2\int_0^{2\pi} -2\sin^2 t + \cos^2 t\,dt$$

$$= R^2\left[\int_0^{2\pi} -2\sin^2 t\,dt + \int_0^{2\pi}\cos^2 t\,dt\right]$$

$$= R^2\left\{\left[\sin(t)\cos(t) - t\right]_0^{2\pi} + \left[\tfrac{1}{2}(t + \cos(t)\sin(t))\right]_0^{2\pi}\right\}$$

$$= R^2\left\{\left[(\sin(2\pi)\cos(2\pi) - 2\pi) - (\sin(0)\cos(0) - 0)\right] + \tfrac{1}{2}\left[(2\pi + \cos(2\pi)\sin(2\pi)) - (0 + \cos(0)\sin(0))\right]\right\}$$

$$= R^2\left[-2\pi + \pi\right] = \boxed{-\pi R^2}$$

$$\iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \iint_{x^2+y^2\le R^2}(1-2)\,dx\,dy = \iint_{x^2+y^2\le R^2} -1\,dx\,dy = -\text{Area(Circle of radius } R) = \boxed{-\pi R^2}$$

(12) Using the divergence theorem, show that $\int_{\partial D}\vec{F}\cdot\hat{n}\,ds = 0$, where $F(x,y) = \langle y, -x\rangle$; D = unit disc.
Verify this directly.

Solution: the divergence theorem states:

F?

$$\int_{\partial D}\vec{F}\cdot\vec{n}\,ds = \iint_D \operatorname{div} F\,dA = \text{in our case} = \iint_{x^2+y^2\le 1}\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\,dA = \iint_{x^2+y^2\le 1} 0 + 0\,dA = \boxed{0}$$

Let us verify this directly:

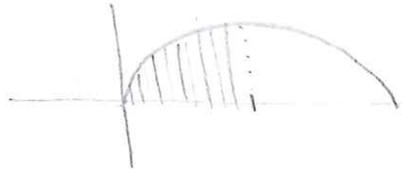
$$\int_{\partial D}\vec{F}\cdot\vec{n}\,ds;\quad \vec{n} = \frac{T_r\times T_\theta}{\|T_r\times T_\theta\|} = \langle x, y\rangle \text{ (normal vector on unit disk).}$$

$$\int_{\partial D}\langle y, -x\rangle\cdot\langle x, y\rangle\,ds = \int_{\partial D} yx - xy\,ds = \int_{\partial D} 0\,ds = \boxed{0}$$

(13) Find the area bounded by one arc of the cycloid
$x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$, where $a>0$ and $0 \le \theta \le 2\pi$, and the x axis.

Use Green's theorem.

Solution: the area we want is:



$$A = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx = \frac{1}{2}\int_{\partial D} a(\theta-\sin\theta)\,d(a-a\cos\theta) - a(1-\cos\theta)\,d(a\theta - a\sin\theta)$$

$$= \frac{1}{2}\int_{\partial D} a(\theta-\sin\theta)(a\sin\theta) - a(1-\cos\theta)(a-a\cos\theta)$$

$$= \frac{1}{2}\int_{\partial D} a^2(\theta\sin\theta - \sin^2\theta) - a^2(1-\cos\theta)^2 = \frac{a^2}{2}\int_{\partial D} \theta\sin\theta - \sin^2\theta - 1+2\cos\theta-\cos^2\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} \theta\sin\theta + 2\cos\theta - 1 - \overbrace{(\sin^2\theta+\cos^2\theta)}^{1}\,d\theta = \frac{a^2}{2}\int_0^{2\pi}\theta\sin\theta + 2\cos\theta - 2\,d\theta$$

$$= \frac{a^2}{2}\Big[\sin\theta - \theta\cos(\theta) + 2\sin\theta - 2\theta\Big]_0^{2\pi} = \frac{a^2}{2}\left[0 - 2\pi + 0 - 4\pi\right] = \frac{a^2}{2}(-6\pi)$$
$$= -3\pi a^2$$

The area is the abs. value : $\boxed{A = 3\pi a^2}$

(the curve should have been oriented counterclockwise).

(18) Let D be a region for which Green's theorem holds. Suppose f is harmonic, i.e., $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ on D.

Prove that : $$\int_{\partial D} \frac{\partial f}{\partial y}dx - \frac{\partial f}{\partial x}dy = 0$$

Pf: By Green's theorem:

where, $P = \frac{\partial f}{\partial y}$, $Q = -\frac{\partial f}{\partial x}$

$$\int_{\partial D}\frac{\partial f}{\partial y}dx + \left(-\frac{\partial f}{\partial x}dy\right) = \iint_D -\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)dA;$$

$$= \iint_D -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\,dA$$

By assumption $\frac{\partial^2 f}{\partial x^2} = -\frac{\partial^2 f}{\partial y^2}$

$$= \iint_D \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial y^2}\,dA = \iint_D 0\,dA = \boxed{0}$$

(19) (a) Verify the divergence theorem for $F = \langle x, y \rangle$ and D the unit disk $x^2+y^2 \leq 1$.

Solution: the divergence theorem states

$$\int_{\partial D} \vec{F} \cdot \hat{n}\, ds = \iint_D \operatorname{div} F\, dA$$; Let us compute each side seperately:

$$\iint_D \operatorname{div} F\, dA = \iint_{x^2+y^2\leq 1} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\, dA = \iint_{x^2+y^2\leq 1} 1+1\, dA = 2\iint_{x^2+y^2\leq 1} dA = 2 \cdot \text{Area unit circle} = \boxed{2\pi}$$

Now the left hand side

$\int_{\partial D} \vec{F} \cdot \hat{n}\, ds$, note that $\hat{n} = \langle x, y \rangle$; since D is a unit disk.
Hence $\vec{F} \cdot \hat{n} = \langle x, y\rangle \cdot \langle x, y \rangle = x^2+y^2 = 1$.

therefore,

$$\int_{\partial D} \vec{F} \cdot \hat{n}\, ds = \int_{\partial D} 1 \cdot ds = \text{circumference of unit circle} = \boxed{2\pi}$$

So the divergence th. is verified for this particular case.

(b) Evaluate the integral of the normal component of $\langle 2xy, -y^2 \rangle$ around the ellipse defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: $$\int_{\partial D} \vec{F} \cdot \hat{n}\, ds = \iint_D \operatorname{div} F\, dA$$

$$= \iint_D 2y - 2y\, dA$$

$$= \iint_D 0\, dA = \boxed{0}$$