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(1) Exercise 7.6.2. Evaluate the surface integral

$\iint_S \vec{F} \cdot d\vec{S}$; where $F(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z^2\hat{k}$ and S is the surface parametrized by $\Phi(u,v) = (2\sin u, 3\cos u, v)$, $0 \le u \le 2\pi$, $0 \le v \le 1$.

Solution: By definition:

$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (T_u \times T_v)\,du\,dv$, where $D = [0,2\pi] \times [0,1]$. and

$T_u = \langle 2\cos u, -3\sin u, 0\rangle$; $T_v = \langle 0,0,1\rangle$. $T_u \times T_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2\cos u & -3\sin u & 0 \\ 0 & 0 & 1 \end{vmatrix} =$

$= \hat{\imath}(-3\sin u) - \hat{\jmath}(2\cos u) + \hat{k}(0) = \langle -3\sin u, -2\cos u, 0\rangle$.

$\vec{F} \cdot (T_u \times T_v) = \langle x, y, z^2\rangle \cdot \langle -3\sin u, -2\cos u, 0\rangle = \langle 2\sin u, 3\cos u, v^2\rangle \cdot \langle -3\sin u, -2\cos u, 0\rangle$

$= -6\sin^2 u - 6\cos^2 u + v^2 \cdot 0 = -6(\sin^2 u + \cos^2 u) = -6$. Hence,

$\iint_D \vec{F} \cdot (T_u \times T_v)\,du\,dv = \int_0^1 \int_0^{2\pi} -6\,du\,dv = \boxed{-12\pi}$

(2) Exercise 7.6.5. Let the temperature of a point in $\mathbb{R}^3$ be given by $T(x,y,z) = 3x^2 + 3z^2$. Compute the heat flux across the surface $x^2 + z^2 = 2$, $0 \le y \le 2$, if $k = 1$.

Solution: By definition, heat "flows" with the vector field: $-k\nabla T = F \Rightarrow$ $F = -k\langle 6x, 0, 6z\rangle$; where $k=1$ we get $F = \langle -6x, 0, -6z\rangle$. We wish to compute the flux, i.e.: $\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (T_u \times T_v)\,du\,dv$. Now we need a parametrization for our surface. Consider:

$\Phi(u,v) = \langle \sqrt{2}\cos u, v, \sqrt{2}\sin u\rangle$; $0 \le u \le 2\pi$, $0 \le v \le 2$.

$T_u = \langle -\sqrt{2}\sin u, 0, \sqrt{2}\cos u\rangle$; $T_v = \langle 0,1,0\rangle \Rightarrow T_u \times T_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sqrt{2}\sin u & 0 & \sqrt{2}\cos u \\ 0 & 1 & 0 \end{vmatrix} =$

$= \hat{\imath}(-\sqrt{2}\cos u) - \hat{\jmath}(0) + \hat{k}(-\sqrt{2}\sin u) = \sqrt{2}\langle -\cos u, 0, -\sin u\rangle$. therefore,

$\vec{F} \cdot (T_u \times T_v) = \langle -6x, 0, -6z\rangle \cdot \langle -\sqrt{2}\cos u, 0, -\sqrt{2}\sin u\rangle =$

$= \langle -6(\sqrt{2}\cos u), 0, -6(\sqrt{2}\sin u)\rangle \cdot \langle -\sqrt{2}\cos u, 0, -\sqrt{2}\sin u\rangle = 12. \Rightarrow$

$$\iint_D \vec{F}\cdot(T_u \times T_v)\,du\,dv = \int_0^2\int_0^{2\pi} 12\,du\,dv = 12\cdot 2\pi\cdot 2 = \boxed{48\pi}$$

(3) Exercise 7.6.7. Let S be the closed surface that consists of the hemisphere $x^2+y^2+z^2=1$, $z \geq 0$, an its base $x^2+y^2 \leq 1$, $z=0$.
Let $\vec{E}$ be the electric field defined by $\vec{E}(x,y,z) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$.
Find the electric flux across S.

Solution: Let us break S into two pieces:

S_1 = the hemisphere $x^2+y^2+z^2=1$, $z \geq 0$.
S_2 = the base $x^2+y^2 \leq 1$, $z=0$.
the total flux will be $\iint_{S_1} \vec{E}\cdot d\vec{S} + \iint_{S_2} \vec{E}\cdot d\vec{S}$.

(I) $\iint_{S_1} \vec{E}\cdot d\vec{S} = \iint_{S_1} \vec{E}\cdot\vec{n}\,dS$; where $\vec{n} = \langle x,y,z\rangle$

$\vec{E}\cdot\vec{n} = \langle 2x,2y,2z\rangle\cdot\langle x,y,z\rangle = 2x^2+2y^2+2z^2 = 2(x^2+y^2+z^2) = 2$. Hence,

$\iint_{S_1} \vec{E}\cdot d\vec{S} = \iint_S \vec{E}\cdot\vec{n}\,dS = 2\iint_S dS = 2\,A(S) = 2(2\pi) = \boxed{4\pi}$

(II) $\iint_{S_2} \vec{E}\cdot d\vec{S} = \iint_{S_1} \vec{E}\cdot\vec{n}\,dS$; where $\vec{n} = \langle 0,0,1\rangle$.

$\vec{E}\cdot\vec{n} = \langle 2x,2y,2z\rangle\cdot\langle 0,0,1\rangle = 2z$. Hence,

$\iint_{S_2} \vec{E}\cdot d\vec{S} = \iint_{S_2} \vec{E}\cdot\vec{n}\,dS = \iint_{S_2} 2z\,dS = 2\iint_{S_2} z\,dS = 2\iint_D z\cdot\|T_u\times T_v\|\,du\,dv$; where $\Phi(u,v) = \langle v\cos u, v\sin u, 0\rangle$; $0 \leq u \leq 2\pi$; $0 \leq v \leq 1$. But note that with this parametrization we have: $2\iint_D z\,\|T_u\times T_v\|\,du\,dv = 2\iint_D 0\cdot\|T_u\times T_v\|\,du\,dv = 0$

Hence, $\iint_{S_2} \vec{E}\cdot d\vec{S} = \boxed{0}$.

therefore (I) + (II) $\Rightarrow \iint_{S_1} \vec{E}\cdot d\vec{S} + \iint_{S_2} \vec{E}\cdot d\vec{S} = 4\pi + 0 = \boxed{4\pi} \to$ the total flux

(4) Exercise 7.6.10. Evaluate $\iint_S (\nabla \times F) \cdot d\vec{S}$, where $\vec{F} = \langle x^2+y-4, 3xy, 2xz+z^2 \rangle$ and S is the surface $x^2+y^2+z^2=16$, $z \geq 0$. (Let $\vec{n}$, the unit normal, be upward pointing).

Solution: $\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y}(2xz+z^2) - \frac{\partial}{\partial z}(3xy)\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}(2xz+z^2) - \frac{\partial}{\partial z}(x^2+y-4)\right) + \hat{k}\left(\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(x^2+y-4)\right).$

$= \hat{\imath}(0-0) - \hat{\jmath}(2z-0) + \hat{k}(3y-1) = \langle 0, -2z, 3y-1 \rangle = \nabla \times F$

Using spherical coord. $x = 4\cos\theta\sin\varphi$, $y = 4\sin\theta\sin\varphi$, $z = 4\cos\varphi$, $0 \leq \varphi \leq \pi/2$, $0 \leq \theta \leq 2\pi$

$\Rightarrow \vec{n} = 4\sin\varphi\langle x, y, z \rangle$.

$\iint_S (\nabla \times F) \cdot d\vec{S} = \iint_S (\nabla \times F) \cdot \vec{n} \cdot dS$ where $(\nabla \times F) \cdot \vec{n} = \langle 0, -2z, 3y-1 \rangle \cdot \langle x, y, z \rangle (4\sin\varphi)$

$= (-2yz + 3yz - z) 4\sin\varphi$.

Hence,

$\iint_S (\nabla \times F) \cdot \vec{n} \cdot dS = \iint_S 4\sin\varphi(-2yz+3yz-z)\,dS = \int_0^{2\pi}\int_0^{\pi/2} 4\sin\varphi\,[-2(4\sin\theta\sin\varphi)(4\cos\varphi) + 3(4\sin\theta\sin\varphi)(4\cos\varphi) - 4\cos\varphi]\,d\varphi\,d\theta$

$= \int_0^{2\pi}\int_0^{\pi/2} 4\sin\varphi\,[-32\sin\theta\sin\varphi\cos\varphi + 48\sin\theta\sin\varphi\cos\varphi - 4\cos\varphi]\,d\varphi\,d\theta$

$= \int_0^{2\pi}\int_0^{\pi/2} 4\sin\varphi\,[16\sin\theta\sin\varphi\cos\varphi - 4\cos\varphi]\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} 64\sin\theta\sin^2\varphi\cos\varphi - 16\sin\varphi\cos\varphi\,d\varphi\,d\theta$

$= 16\int_0^{2\pi}\int_0^{\pi/2} 4\sin\theta\sin^2\varphi\cos\varphi - \sin\varphi\cos\varphi\,d\varphi\,d\theta = 16\int_0^{2\pi}\left(\frac{4}{3}\sin\theta - \frac{1}{2}\right)d\theta = \boxed{-16\pi}$

(5) Exercise 7.6.13. Find the flux of the vector field $V(x,y,z) = 3xy^2\,\vec{\imath} + 3x^2y\,\vec{\jmath} + z^3\,k$ out of the unit sphere.

Solution: here $\vec{n} = \langle x, y, z \rangle$. So the flux is given by:

$\iint_S (V \cdot \vec{n})\,dS = \iint_S 3x^2y^2 + 3x^2y^2 + z^4\,dS = \iint_S 6x^2y^2 + z^4\,dS$, where S can be parametized using spherical coordinates:

$= \iint_D (6x^2y^2+z^4)\,\|T_u \times T_v\|\,du\,dv$.

$x = \cos\theta\sin\varphi$, $y = \sin\theta\sin\varphi$, $z = \cos\varphi$, $0 \leq \varphi < \pi$, $0 \leq \theta < 2\pi$.

$T_\theta = \langle -\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0 \rangle$; $T_\varphi = \langle \cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi \rangle$;

$$\Phi' = \begin{pmatrix} -\sin\theta\sin\varphi & \cos\theta\sin\varphi & 0 \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \end{pmatrix}$$

$$\left|\frac{\partial(x,y)}{\partial(\theta,\varphi)}\right| = -\sin^2\theta\sin\varphi\cos\varphi - \cos^2\theta\cos\varphi\sin\varphi = -\sin\varphi\cos\varphi(\sin^2\theta+\cos^2\theta) = -\sin\varphi\cos\varphi.$$

$$\left|\frac{\partial(x,z)}{\partial(\theta,\varphi)}\right| = \sin\theta\sin^2\varphi - 0 = \sin\theta\sin^2\varphi.$$

$$\left|\frac{\partial(y,z)}{\partial(\theta,\varphi)}\right| = -\cos\theta\sin^2\varphi = -\cos\theta\sin^2\varphi.$$

$$\|T_\theta\times T_\varphi\| = \sqrt{\sin^2\varphi\cos^2\varphi + \sin^2\theta\sin^4\varphi + \cos^2\theta\sin^4\varphi} = \sqrt{\sin^2\varphi\cos^2\varphi + \sin^4\varphi}$$

$$= \sqrt{\sin^2\varphi(\cos^2\varphi+\sin^2\varphi)} = \sin\varphi.$$

$$\iint_D (6x^2y^2+z^4)\,\|T_\theta\times T_\varphi\|\,d\theta\,d\varphi = \int_0^\pi\int_0^{2\pi}\left[6(\cos^2\theta\sin^2\varphi\sin^2\theta\sin^2\varphi)+\cos^4\varphi\right]\sin\varphi\,d\theta\,d\varphi$$

$$= \int_0^{2\pi}\int_0^\pi 6\cos^2\theta\sin^2\varphi\sin^2\theta\sin^3\varphi + \cos^4\varphi\sin\varphi\,d\varphi\,d\theta.$$ Two parts.

(I) $6\int_0^{2\pi}\cos^2\theta\sin^2\theta\left[\int_0^\pi\sin^5\varphi\,d\varphi\right]d\theta$ and (II) $\int_0^{2\pi}\int_0^\pi\cos^4\varphi\sin\varphi\,d\varphi\,d\theta = 2\pi\int_0^\pi\cos^4\varphi\sin\varphi\,d\varphi.$

(I) $$\int_0^\pi\sin^5\varphi\,d\varphi = \left[-\frac{5}{8}\cos(\varphi)+\frac{5}{48}\cos(3\varphi)-\frac{1}{80}\cos(5\varphi)\right]_0^\pi = \left[+\frac{5}{8}-\frac{5}{48}+\frac{1}{80}\right]-\left[-\frac{5}{8}+\frac{5}{48}-\frac{1}{80}\right]$$

$$= \frac{10}{8}-\frac{10}{48}+\frac{2}{80} = \frac{11}{24}+\frac{2}{80} = \frac{29}{60} \Rightarrow$$

$$6\times\frac{29}{60}\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta = \frac{29}{10}\left[\frac{1}{32}(4\theta-\sin(4\theta))\right]_0^{2\pi} = \frac{29}{320}\left[8\pi\right] = \frac{29}{40}\pi$$

(II) $$2\pi\int_0^\pi\cos^4\varphi\sin\varphi\,d\varphi = 2\pi\left[-\frac{1}{5}\cos^5(\varphi)\right]_0^\pi = \frac{-2}{5}\pi\left[-1-1\right] = \frac{4}{5}\pi.$$

(I) & (II) $\Rightarrow$ the flux = $\boxed{\frac{12\pi}{5}}$

(6) Exercise 7.7.1. the helicoid can be described by:

$$\Phi(u,v) = (u\cos v, u\sin v, bv), \text{ where } b \neq 0.$$

Show that $H=0$ and that $K = -b^2/(b^2+u^2)^2$.

Pf: $H = \dfrac{GL+En-2Em}{2W}$, where:

$T_u = \langle \cos v, \sin v, 0\rangle \quad T_v = \langle -u\sin v, u\cos v, b\rangle$

$E = \|T_u\|^2 = 1$; $F = T_u \cdot T_v = 0$; $G = \|T_v\|^2 = u^2+b^2$

$W = EG-F^2 = 1\cdot(u^2+b^2) - 0^2 = u^2+b^2$

$N = \dfrac{T_u \times T_v}{\|T_u \times T_v\|}$; $T_u \times T_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & b \end{vmatrix} = \hat{i}(b\sin v) - \hat{j}(b\cos v) + \hat{k}(u)$
$= \langle b\sin v, -b\cos v, u\rangle$

$\|T_u \times T_v\| = \sqrt{b^2+u^2} \Rightarrow N = \dfrac{\langle b\sin v, -b\cos v, u\rangle}{\sqrt{b^2+u^2}}$

$L = N\cdot T_{uu} = \dfrac{\langle b\sin v, -b\cos v, u\rangle}{\sqrt{b^2+u^2}} \cdot \langle 0,0,0\rangle = 0$

$m = N\cdot T_{uv} = \dfrac{\langle b\sin v, -b\cos v, u\rangle}{\sqrt{b^2+u^2}} \cdot \langle -\sin v, \cos v, 0\rangle = \dfrac{-b}{\sqrt{b^2+u^2}}$

$n = N\cdot T_{vv} = \dfrac{\langle b\sin v, -b\cos v, u\rangle}{\sqrt{b^2+u^2}} \cdot \langle -u\cos v, -u\sin v, 0\rangle = 0$

$$H = \frac{GL+En-2Em}{2W} = \frac{(u^2+b^2)\cdot 0 + 1(0) - 2(0)\left[\frac{-b}{\sqrt{b^2+u^2}}\right]}{2(u^2+b^2)} = \boxed{0}$$

$$K = \frac{Ln-m^2}{W} = \frac{(0)\cdot(0) - \frac{b^2}{b^2+u^2}}{u^2+b^2} = \boxed{\frac{-b^2}{(u^2+b^2)^2}}$$

(7) Exercise 7.7.2. Consider the saddle surface $z = xy$. Show that

$$K = \frac{-1}{(1+x^2+y^2)^2}, \quad H = \frac{-xy}{(1+x^2+y^2)^{3/2}}$$

Pf: First, let us parametrize the saddle surface $z = xy$ as follow:

$$\Phi(u,v) = \langle u, v, uv \rangle.$$

$T_u = \langle 1, 0, v \rangle$; $T_v = \langle 0, 1, u \rangle$; $T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = \hat{i}(-v) - \hat{j}(u) + \hat{k}(1)$
$= \langle -v, -u, 1 \rangle.$

$\|T_u \times T_v\| = \sqrt{v^2+u^2+1}.$

$E = \|T_u\|^2 = 1+v^2$; $F = T_u \cdot T_v = uv$; $G = \|T_v\|^2 = 1+u^2$

$W = EG - F^2 = \|T_u \times T_v\|^2 = v^2+u^2+1.$

$$N = \frac{T_u \times T_v}{\|T_u \times T_v\|} = \frac{\langle -v, -u, 1 \rangle}{\sqrt{v^2+u^2+1}}$$

$$l = N \cdot T_{uu} = \frac{\langle -v, -u, 1 \rangle}{\sqrt{v^2+u^2+1}} \cdot \langle 0, 0, 0 \rangle = 0$$

$$m = N \cdot T_{uv} = \frac{\langle -v, -u, 1 \rangle}{\sqrt{v^2+u^2+1}} \cdot \langle 0, 0, 1 \rangle = \frac{1}{\sqrt{v^2+u^2+1}}$$

$$n = N \cdot T_{vv} = \frac{\langle -v, -u, 1 \rangle}{\sqrt{v^2+u^2+1}} \cdot \langle 0, 0, 0 \rangle = 0$$

$$K = \frac{ln - m^2}{W} = \frac{0 \cdot 0 - \frac{1}{v^2+u^2+1}}{v^2+u^2+1} = \boxed{-\frac{1}{(v^2+u^2+1)^2}}$$

$$H = \frac{Gl + En - 2Fm}{2W} = \frac{(1+u^2)(0) + (1+v^2)(0) - 2(uv)\frac{1}{\sqrt{v^2+u^2+1}}}{2(v^2+u^2+1)} = \boxed{\frac{-uv}{(1+u^2+v^2)^{3/2}}}$$

(8) Exercise 7.7.6. Compute the Gauss curvature of the ellipsoid.

Solution: $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$.

Gauss curvature: $K = \frac{Ln - m^2}{W}$

$\Phi(\varphi, \theta) = \langle a\cos\theta\cos\varphi, a\sin\theta\cos\varphi, c\sin\varphi \rangle$. $\quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \pi$

This parametrization works b/c:

$$\frac{\cancel{a^2}\cos^2\theta\cos^2\varphi}{\cancel{a^2}} + \frac{\cancel{a^2}\sin^2\theta\cos^2\varphi}{\cancel{a^2}} + \frac{\cancel{c^2}\sin^2\varphi}{\cancel{c^2}} = \cos^2\varphi(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + \sin^2\varphi$$
$$= \cos^2\varphi + \sin^2\varphi = \boxed{1}.$$

$T_\varphi = \langle -a\cos\theta\sin\varphi, -a\sin\theta\sin\varphi, c\cos\varphi \rangle$; $T_\theta = \langle -a\sin\theta\cos\varphi, a\cos\theta\cos\varphi, 0 \rangle$

$$T_\varphi \times T_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\cos\theta\sin\varphi & -a\sin\theta\sin\varphi & c\cos\varphi \\ -a\sin\theta\cos\varphi & a\cos\theta\cos\varphi & 0 \end{vmatrix} = \hat{i}(-ac\cos\theta\cos^2\varphi) - \hat{j}(-ac\sin\theta\cos^2\varphi)$$
$$+ \hat{k}(-a^2\cos^2\theta\sin\varphi\cos\varphi - a^2\sin^2\theta\cos\varphi\sin\varphi)$$
$$= \langle -ac\cos\theta\cos^2\varphi, ac\sin\theta\cos^2\varphi, -a^2\sin\varphi\cos\varphi \rangle.$$

$$\|T_\varphi \times T_\theta\| = \sqrt{(ac)^2\cos^2\theta\cos^4\varphi + (ac)^2\sin^2\theta\cos^4\varphi + a^4\sin^2\varphi\cos^2\varphi}$$
$$= \sqrt{(ac)^2\cos^4\varphi + a^4\sin^2\varphi\cos^2\varphi} = \sqrt{\cos^2\varphi((ac)^2\cos^2\varphi + a^4\sin^2\varphi)}$$

$$N = \frac{\langle -ac\cos\theta\cos^2\varphi, ac\sin\theta\cos^2\varphi, -a^2\sin\varphi\cos\varphi \rangle}{\sqrt{\cos^2\varphi((ac)^2\cos^2\varphi + a^4\sin^2\varphi)}}$$

$$L = N \cdot T_{\varphi\varphi} = \frac{\langle -ac\cos\theta\cos^2\varphi, ac\sin\theta\cos^2\varphi, -a^2\sin\varphi\cos\varphi \rangle}{\sqrt{\cos^2\varphi((ac)^2\cos^2\varphi + a^4\sin^2\varphi)}} \cdot \langle -a\cos\theta\cos\varphi, -a\sin\theta\cos\varphi, -c\sin\varphi \rangle$$

$$= \frac{-a^2c\cos^2\theta\cos^3\varphi - a^2c\sin^2\theta\cos^3\varphi + a^2c\sin^2\varphi\cos\varphi}{\sqrt{\cos^2\varphi((ac)^2\cos^2\varphi + a^4\sin^2\varphi}}$$

$$m = N \cdot T_{\varphi\theta} = \frac{\langle -ac\cos\theta\cos^2\varphi, ac\sin\theta\cos^2\varphi, -a^2\sin\varphi\cos\varphi \rangle}{\sqrt{\cos^2\varphi((ac)^2\cos^2\varphi + a^4\sin^2\varphi)}} \cdot \langle a\sin\theta\sin\varphi, -a\cos\theta\sin\varphi, 0 \rangle$$

$$= \frac{-a^2c\cos\theta\cos^2\varphi\sin\theta\sin\varphi - a^2c\sin\theta\cos\theta\cos^2\varphi\sin\varphi}{\sqrt{\cos^2\varphi((ac)^2\cos^2\varphi + a^4\sin^2\varphi}} = \frac{-2a^2c\cos\theta\cos^2\varphi\sin\theta\sin\varphi}{\sqrt{\cos^2\varphi((ac)^2\cos^2\varphi + a^4\sin^2\varphi}}$$

$$n = N\cdot T_{\theta\theta} = \frac{\langle -ac\cos\theta\cos^2\varphi,\ ac\sin\theta\cos^2\varphi,\ -a^2\sin\varphi\cos\varphi\rangle}{\sqrt{\cos^2\varphi((ac)^2\cos^2\varphi + a^4\sin^2\varphi)}}\cdot\langle -a\cos\theta\cos\varphi, -a\sin\theta\cos\varphi, 0\rangle$$

$$= \frac{a^2c\cos^2\theta\cos^3\varphi - a^2c\sin^2\theta\cos^3\varphi}{\sqrt{\cos^2\varphi((ac)^2\cos^2\varphi + a^4\sin^2\varphi)}}$$

What is b?

$$K = \frac{Ln-m^2}{W} = \boxed{\frac{a^2 b^2 c^2}{[a^2b^2\cos^2\varphi + c^2(b^2\cos^2\theta + a^2\sin^2\varphi)\sin^2\varphi]^2}}$$

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(9) Exercise 7.7.7. Integrate

$$\frac{1}{2\pi}\iint_S K\,dA = 2.$$

Solution: Using a different parametization we can compute this integral as

$$\frac{1}{2\pi}\iint_S K\,dA = \int_0^{\pi}\int_0^{2\pi}\frac{a^4c^2\sin v}{(a^4\cos^2 v + a^2c^2\sin^2 v)^{3/2}}\,du\,dv = 2\pi ac^2\int_0^{\pi}\frac{\sin v}{(a^2\cos^2 v + c^2(1-\cos^2 v))^{3/2}}\,dv$$

$$= \frac{2\pi ac^2}{(a^2-c^2)^{\frac{3}{2}}}\int_0^{\pi}\frac{\sin v}{\left(\cos^2 v + \frac{c^2}{a^2-c^2}\right)^{3/2}}\,dv = \frac{2\pi ac^2}{(a^2-c^2)^{3/2}}\int_{-1}^{1}\frac{1}{\left((w)^2 + \left(\sqrt{\frac{c^2}{a^2-c^2}}\right)^2\right)^{3/2}}\,dw$$ by

trigonometric substitution we get $\leadsto$ $\frac{1}{2\pi}\iint_S K\,dA = \frac{1}{2\pi}(4\pi) = \boxed{2}$

(10) Exercise 7.7.10. A parametrization of the torus is:

$$\Phi(\varphi,\theta) = ((R+r\cos\varphi)\cos\theta, (R+r\cos\varphi)\sin\theta, r\sin\varphi); \quad 0\le\varphi\le 2\pi,\ 0\le\theta\le 2\pi$$

$T_\varphi = (-r\sin\varphi\cos\theta, -r\sin\varphi\sin\theta, r\cos\varphi)$. $T_\theta = (-(R+r\cos\varphi)\sin\theta, (R+r\cos\varphi)\cos\theta, 0)$.

$T_\varphi\times T_\theta = -r(R+r\cos\varphi)\langle\cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi\rangle$. $\Rightarrow \|T_\varphi\times T_\theta\| = r(R+r\cos\varphi)$.

$N = -\langle\cos\varphi\cos\theta, \cos\varphi\cos\theta, \sin\varphi\rangle$

$E = \|T_\varphi\|^2 = r^2$; $F = T_\varphi\cdot T_\theta = 0$; $G = \|T_\theta\|^2 = (R+\cos\varphi)^2$

$\Phi_{\varphi\varphi} = -r\langle\cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi\rangle$; $\Phi_{\varphi\theta} = r\langle\sin\varphi\sin\theta, -\sin\varphi\cos\theta, 0\rangle$

$\Phi_{\theta\theta} = -(R+r\cos\varphi)\langle\cos\theta, \sin\theta, 0\rangle$.

$L = N\cdot\Phi_{\varphi\varphi} = r$; $m = N\Phi_{\varphi\theta} = 0$; $n = N\cdot\Phi_{\theta\theta} = (R+r\cos\varphi)\cos\varphi$.

$W = EG - F^2 = r^2(R+r\cos\varphi)^2$ Finally, the Gauss curvature K is:

$$K = \frac{\cos\varphi}{r(R+r\cos\varphi)}; \text{ so if } r=1 \Rightarrow \boxed{K = \frac{\cos\varphi}{R+\cos\varphi}}$$

Verification of Gauss-Bonnet:

$$\frac{1}{2\pi}\iint_S \kappa\, dA = \frac{1}{2\pi}\iint_D \frac{\cos\varphi}{R+\cos\varphi}\, dA = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\cos\varphi}{R+\cos\varphi} \|T_\varphi \times T_\theta\|\, d\varphi\, d\theta.$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\cos\varphi}{\cancel{R+\cos\varphi}} (R+\cos\varphi)\, d\varphi\, d\theta = \frac{2\pi}{2\pi}\int_0^{2\pi} \cos\varphi\, d\varphi = \boxed{0};$$

which is true since the torus has genus 2 and by the Gauss Bonnet theorem $\frac{1}{2\pi}\iint_S \kappa\, dA = 2-2g$; $g=1 \Rightarrow \frac{1}{2\pi}\iint_S \kappa\, dA = \boxed{0}$