M343 Homework 7

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June 05, 2013

Section 4.2

16. Consider the equation $y^{(4)} - 5y'' + 4y = 0$. The solution is given by solving the characteristic equation:

$$r^4 - 5r^2 + 4 = 0$$

It is not hard to see (or using the rational root theorem) that 1 is a root of this equation, and so we can divide the equation by $r-1$ to obtain: $r^4 - 5r^2 + 4 = 0 \iff (r-1)(r^3 + r^2 - 4r - 4) = 0$. Likewise, is is not hard to see that -2 is a root of $(r^3 + r^2 - 4r - 4)$ and so, by polynomial division $(r-1)(r^3 + r^2 - 4r - 4) = 0 \iff (r-1)(r+2)(r^2 - r - 2) = 0 \iff (r-1)(r+2)(r+1)(r-2) = 0$. The solution is therefore:

$$\boxed{y = C_1 e^t + C_2 e^{-2t} + C_3 e^{-t} + C_4 e^{2t}}$$

21. Consider the equation $y^{(8)} + 8y^{(4)} + 16y = 0$. The solution is given by solving the characteristic equation:

$$r^8 + 8r^4 + 16 = 0 \iff (r^4 + 4)^2 = 0 \iff r_{1,2,\cdots,8} = \pm 1 + i$$

The solution is therefore:

$$\boxed{y = e^t[(C_1 + C_2 t)cos(t) + (C_3 + C_4 t)sin(t)] + e^{-t}[(C_5 + C_6 t)cos(t) + (C_7 + C_8 t)sin(t)]}$$

29. Consider the initial value problem $y''' + y' = 0;\ \ y(0) = 0;\ \ y'(0) = 1;\ \ y''(0) = 2$.
The solution is given by solving the characteristic equation:

$$r^3 + r = 0 \iff r(r^2 + 1) = 0 \Longrightarrow r_1 = 0;\quad r_{2,3} = \pm i$$

The general solution is therefore:

$$\boxed{y = C_1 + C_2 cos(t) + C_3 sin(t)}$$

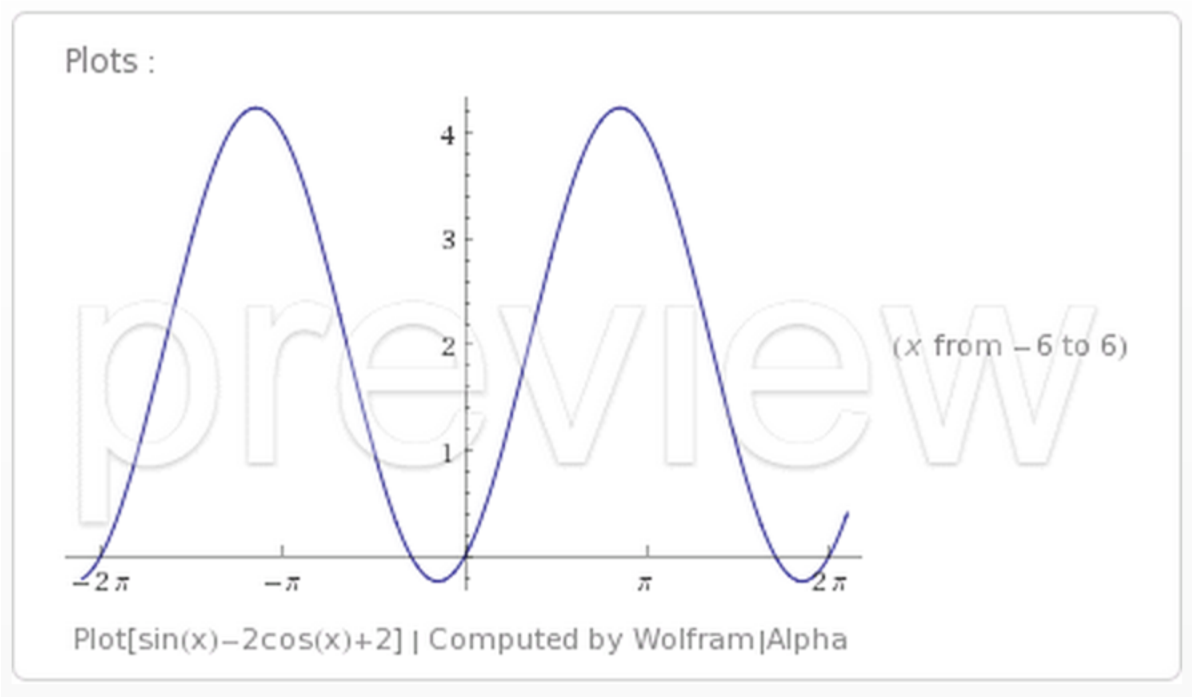
The solution to the I.V.P is given by solving the system of linear equations:

$$\begin{cases} y(0) = 0 = C_1 + C_2 \Longrightarrow C_1 = -C_2 \Longrightarrow \boxed{C_1 = 2} \\ \\ y'(0) = 1 = -C_2 sin(0) + C_3 cos(0) \Longrightarrow \boxed{C_3 = 1} \\ \\ y''(0) = 2 = -C_2 cos(0) - C_3 sin(0) \Longrightarrow \boxed{C_2 = -2} \\ \end{cases}$$

The solution to the I.V.P is:

$$\boxed{y = 2 - 2cos(t) + sin(t)}$$

The graph of the solution is:



The solution oscillates as $t \to \infty$

32. Consider the initial value problem $y''' - y'' + y' - y = 0$; $\;y(0) = 2$; $\;y'(0) = -1$; $\;y''(0) = -2$.
The solution is given by solving the characteristic equation:

$$r^3 - r^2 + r - 1 = 0$$

It is not hard to see (or using the rational root theorem) that 1 is a root of this equation, and so we can divide the equation by $r - 1$ to obtain:

$$r^3 - r^2 + r - 1 = 0 \iff (r-1)(r^2+1) = 0 \Longrightarrow r_1 = 1;\; r_{2,3} = \pm i$$

The general solution is therefore:

$$\boxed{y = C_1 e^t + C_2 cos(t) + C_3 sin(t)}$$

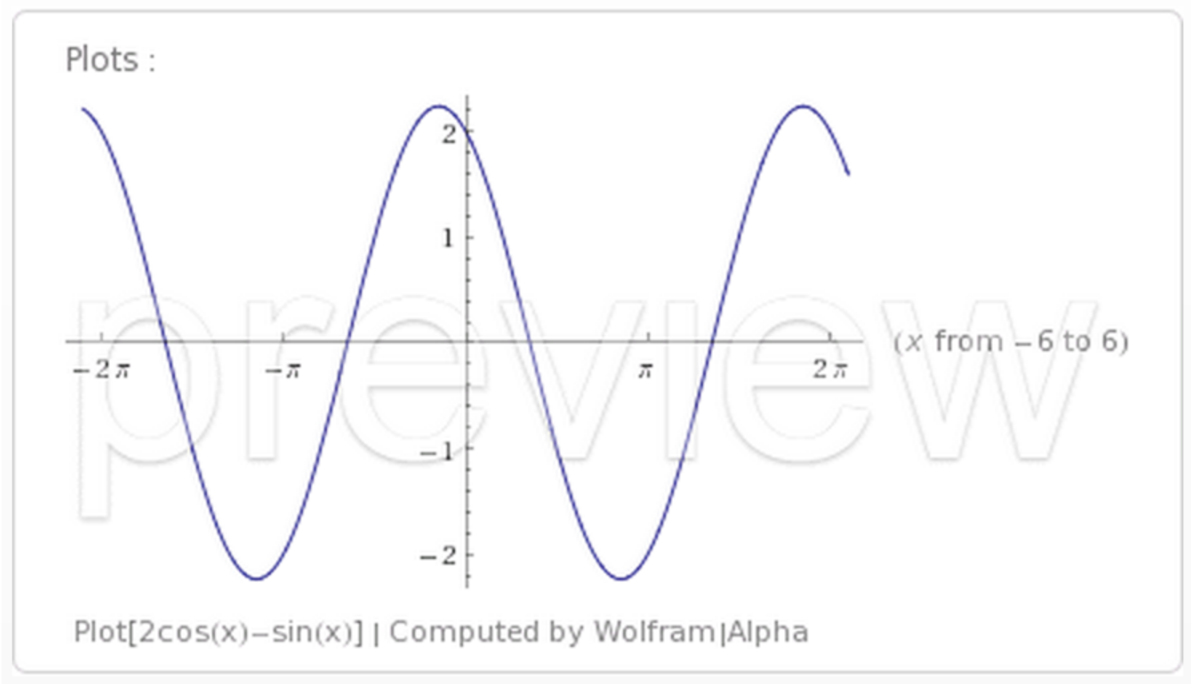
The solution to the I.V.P is given by solving the system of linear equations:

$$\begin{cases} y(0) = 2 = C_1 + C_2 \Longrightarrow C_1 = -C_2 \Longrightarrow \boxed{C_2 = 2} \\ \\ y'(0) = -1 = C_1 - C_2 sin(0) + C_3 cos(0) \Longrightarrow -1 = C_1 + C_3 \Longrightarrow \boxed{C_1 = 0} \\ \\ y''(0) = -2 = C_1 - C_2 cos(0) - C_3 sin(0) \Longrightarrow -2 = C_1 - C_2 \Longrightarrow \boxed{C_3 = -1} \\ \end{cases}$$

The solution to the I.V.P is:

$$\boxed{y = 2cos(t) - sin(t)}$$

The graph of the solution is:



The solution oscillates as $t \to \infty$

Section 4.3

10. Consider the initial value problem $y^{(4)} + 2y'' + y = 3t + 4$; $y(0) = 0$; $y'(0) = 0$; $y''(0) = y'''(0) = 1$. The solution is given by

$$y_g = y_h + y_p, \quad \text{where,}$$

$\underline{y_h}$ is given by solving the characteristic equation:

$$r^4 + 2r^2 + 1 = 0 \iff (r^2+1)^2 = 0 \Longrightarrow r_{1,2} = \pm i; \quad r_{3,4} = \pm i$$

The homogeneous solution is:

$$\boxed{y_h = C_1 cos(t) + C_2 sin(t) + C_3 t cos(t) + C_4 t sin(t)}$$

$\underline{y_p}$ guess for particular solution: $y_p = At + B$. Then, $y_p' = A$ and $y_p'' = y_p''' = y^{(4)} = 0$. Suppose y_p satisfies:

$$y_p^{(4)} + 2y_p'' + y_p = 3t + 4 \iff 0 + 2 \cdot 0 + At + B = 3t + 4 \Longrightarrow \boxed{A = 3}; \ \boxed{B = 4}$$

The general solution is therefore:

$$\boxed{y = C_1 cos(t) + C_2 sin(t) + C_3 t cos(t) + C_4 t sin(t) + 3t + 4}$$

Solving for C_1, C_2, C_3, C_4;

$$\begin{cases} y(0) = 0 = C_1 + 4 \Longrightarrow \boxed{C_1 = -4} \\ y'(0) = 0 = -C_1 sin(0) + C_2 cos(0) + C_3 cos(0) - C_3 t sin(0) + C_4 sin(0) + C_4 t cos(0) + 3 \Longrightarrow \boxed{C_4 = -\frac{3}{2}} \\ y''(0) = 1 = -cos(0) - C_2 sin(0) - 2C_3 sin(0) - C_3 t cos(0) + 2C_4 cos(0) - C_4 t sin(0) \Longrightarrow \boxed{C_3 = 1} \\ y'''(0) = 1 = C_1 sin(0) - C_2 cos(0) - 3C_3 cos(0) + C_3 t sin(0) - 3C_4 sin(0) - C_4 t sin(0) \Longrightarrow \boxed{C_2 = -4} \end{cases}$$

The solution to the I.V.P is:

$$\boxed{y = -4cos(t) - 4sin(t) + tcos(t) - \frac{3}{2}tsin(t) + 3t + 4}$$

11. Consider the initial value problem $y''' - 3y'' + 2y' = t + e^t;\ \ y(0) = 1;\ \ y'(0) = -\frac{1}{4};\ \ y''(0) = -\frac{3}{2}$.
The solution is given by

$$y_g = y_h + y_p, \quad \text{where,}$$

$\underline{y_h}$ is given by solving the characteristic equation:

$$r^3 - 3r^2 + 2r = 0 \iff r(r-1)(r-2) = 0 \Longrightarrow r_1 = 0;\quad r_2 = 1;\quad r_3 = 2$$

The homogeneous solution is:

$$\boxed{y_h = C_1 + C_2e^t + C_3e^{2t}}$$

$\underline{y_p}$ guess for particular solution: $y_p = t(At + B) + Cte^t$. Then:

$$y_p' = 2At + B + Ce^t + Cte^t$$

$$y_p'' = 2A + 2Ce^t + Cte^t$$

$$y_p''' = 3Ce^t + Cte^t$$

Suppose y_p satisfies:

$$\begin{aligned}
t + e^t &= y_p''' - 3y_p'' + 2y_p' \\
&= 3Ce^t + Cte^t - 3(2A + 2Ce^t + Cte^t) + 2(2At + B + Ce^t + Cte^t) \\
&= e^t(3C - 6C + 2C) + te^t(C - 3C + 2C) + 4At + 2B - 6A \\
&= -Ce^t + 4At + 2B - 6A
\end{aligned}$$

Therefore, $A - \frac{1}{4}, B = \frac{3}{4}, C = -1$ The general solution is therefore:

$$\boxed{y = C_1 + C_2e^t + C_3e^{2t} + \frac{1}{4}t^2 + \frac{3}{4}t - te^t}$$

Solving for C_1, C_2, C_3;

$$\begin{cases}
y(0) = 1 = C_1 + C_2 + C_3 \Longrightarrow \boxed{C_1 = 1} \\
y'(0) = -\frac{1}{4} = C_2 + 2C_3 - \frac{1}{4} \Longrightarrow \boxed{C_3 = 0} \\
y''(0) = -\frac{3}{2} = C_2 + 4C_3 - \frac{3}{2} \Longrightarrow \boxed{C_2 = 0}
\end{cases}$$

The solution to the I.V.P is:

$$\boxed{y = \frac{1}{4}(t^2 + 3t) - te^t + 1}$$

14. Consider the equation $y''' - y' = te^{-t} + 2cos(t)$. Solving the characteristic equation:

$$r^3 - r = 0 \iff r(r^2 - 1) = 0 \Longrightarrow r_1 = 0;\ r_2 = 1;\ r_3 = -1$$

So the homogeneous solution is

$$\boxed{y_h = C_1 + C_2e^t + C_3e^{-t}}$$

A guess for the particular solution y_p is:

$$\boxed{y_p = (At^2 + Bt)e^{-t} + Ccos(t) + Dsin(t)}$$

15. Consider the equation $y^{(4)} - 2y'' + y = e^t + sin(t)$. Solving the characteristic equation:

$$r^4 - 2r^2 + 1 = 0 \iff (r^2 - 1)^2 = 0 \Longrightarrow r_1 = 1;\ r_2 = -1;\ r_3 = 1;\ r_4 = -1$$

So the homogeneous solution is

$$\boxed{y_h = C_1 e^t + C_2 e^{-t} + C_3 t e^t + C_4 t e^{-t}}$$

A guess for the particular solution y_p is:

$$\boxed{y_p = At^2 e^t + Bsin(t) + Ccos(t)}$$

Section 4.4

1. $y''' + y' = tan(t)$
The solution by Variation of Parameters is given by

$$y_g = u_1 y_1 + u_2 y_2 + u_3 y_3, \quad \text{where,}$$

y_1, y_2, y_3 are found by solving the homogeneous eq. In turn, y_h is given by solving the characteristic equation:

$$r^3 + r = 0 \iff r(r^2 + 1) = 0 \Longrightarrow r_1 = 0; \quad r_{2,3} = \pm i$$

The homogeneous solution is:

$$\boxed{y_h = C_1 + C_2 cos(t) + C_3 sin(t)}$$

Let $y_1 = e^t$; $y_2 = cos(t)$; $y_3 = sin(t)$. Then, as previously calculated:

$$W(1, cos(t), sin(t)) = \begin{vmatrix} 1 & cos(t) & sin(t) \\ 0 & -sin(t) & cos(t) \\ 0 & -cos(t) & -sin(t) \end{vmatrix} =$$

$$\begin{vmatrix} -sin(t) & cos(t) \\ -cos(t) & -sin(t) \end{vmatrix} - cos(t) \begin{vmatrix} 0 & cos(t) \\ 0 & -sin(t) \end{vmatrix} + sin(t) \begin{vmatrix} 0 & -sin(t) \\ 0 & -cos(t) \end{vmatrix} = sin^2(t) + cos^2(t) = \boxed{1}$$

$$W_1 = 1; \quad W_2 = -cos(t); \quad W_3 = -sin(t)$$

$$u_1' = \frac{W_1 \cdot g}{W} = tan(t) \Longrightarrow \boxed{u_1 = -ln(|cos(t)|) + C_1}$$

$$u_2' = \frac{W_2 \cdot g}{W} = -sin(t) \Longrightarrow \boxed{u_2 = cos(t) + C_2}$$

$$u_3' = \frac{W_3 \cdot g}{W} = -sin(t)tan(t) \Longrightarrow \text{(trigonometric substitution...)} \boxed{u_3 = sin(t) - ln(|sec(t) + tan(t)|) + C_3}$$

Therefore

$$y_g = u_1 y_1 + u_2 y_2 + u_3 y_3 \iff \boxed{y = C_1 + C_2 cos(t) + C_3 sin(t) - ln(|cos(t)|) - sin(t) ln(|sec(t) + tan(t)|)}$$

5. $y'' - y'' + y' - y = e^{-t} sin(t)$
The solution by Variation of Parameters is given by

$$y_g = u_1 y_1 + u_2 y_2 + u_3 y_3, \quad \text{where,}$$

y_1, y_2, y_3 are found by solving the homogeneous eq. In turn, y_h is given by solving the characteristic equation:

$$r^3 - r^2 + r - 1 = 0 \iff (r - 1)(r^2 + 1) = 0 \Longrightarrow r_1 = 0; \quad r_{2,3} = \pm i$$

The homogeneous solution is:

$$\boxed{y_h = C_1 e^t + C_2 cos(t) + C_3 sin(t)}$$

Let $y_1 = e^t$; $y_2 = cos(t)$; $y_3 = sin(t)$. Then, as previously calculated:

$$W(e^t, cos(t), sin(t)) = \begin{vmatrix} e^t & cos(t) & sin(t) \\ e^t & -sin(t) & cos(t) \\ e^t & -cos(t) & -sin(t) \end{vmatrix} =$$

$$e^t\begin{vmatrix}-sin(t) & cos(t)\\ -cos(t) & -sin(t)\end{vmatrix} - cos(t)\begin{vmatrix}e^t & cos(t)\\ e^t & -sin(t)\end{vmatrix} + sin(t)\begin{vmatrix}e^t & -sin(t)\\ e^t & -cos(t)\end{vmatrix} = \boxed{2e^t}$$

$$W_1 = 1; \quad W_2 = e^t(sin(t) - cos(t)); \quad W_3 = e^t(cos(t) - sin(t))$$

$$u_1' = \frac{W_1 \cdot g}{W} = \frac{sin(t)}{2e^{2t}} \Longrightarrow \text{(integration by parts twice...)} \boxed{u_1 = -\frac{1}{10}[e^{-2t}(2sin(t) + cos(t))] + C_1}$$

$$u_2' = \frac{W_2 \cdot g}{W} = \frac{sin^2(t) - sin(t)cos(t)}{2e^t} \Longrightarrow \boxed{u_2 = \frac{1}{20e^t}[-sin(2t) + 3cos(2t) - 5] + C_2}$$

$$u_3' = \frac{W_3 \cdot g}{W} = -tan(t) \Longrightarrow \boxed{u_3 = -\frac{1}{20e^t}[-sin(2t) + 3cos(2t) - 5] + C_3}$$

Therefore

$$y_g = u_1y_1 + u_2y_2 + u_3y_3 \iff \boxed{y = C_1e^t + C_2cos(t) + C_3sin(t) - \frac{1}{5}e^{-t}(sin(t) + cos(t))}$$

9. $y''' + y' = sec(t), \quad y(0) = 2; \; y'(0) = 1; \; y''(0) = -2$. The solution by Variation of Parameters is given by

$$y_g = u_1y_1 + u_2y_2 + u_3y_3, \quad \text{where,}$$

y_1, y_2, y_3 are found by solving the homogeneous eq. In turn, $\underline{y_h}$ is given by solving the characteristic equation:

$$r^3 + r = 0 \iff r(r^2 + 1) = 0 \Longrightarrow r_1 = 0; \quad r_{2,3} = \pm i$$

The homogeneous solution is:

$$\boxed{y_h = C_1 + C_2cos(t) + C_3sin(t)}$$

Let $y_1 = 1; \quad y_2 = cos(t); \quad y_3 = sin(t)$. Then, as previously calculated:

$$W(1, cos(t), sin(t)) = 1; \; W_1 = 1; \; W_2 = -cos(t); \; W_3 = -sin(t)$$

$$u_1' = \frac{W_1 \cdot g}{W} = sec(t) \Longrightarrow \boxed{u_1 = ln(|sec(t) + tan(t)|) + C_1}$$

$$u_2' = \frac{W_2 \cdot g}{W} = -cos(t)\frac{1}{cos(t)} = -1 \Longrightarrow \boxed{u_2 = -t + C_2}$$

$$u_3' = \frac{W_3 \cdot g}{W} = -tan(t) \Longrightarrow \boxed{u_3 = ln(|cos(t)|) + C_3}$$

Therefore

$$y_g = u_1y_1 + u_2y_2 + u_3y_3 \iff \boxed{y = C_1 + C_2cos(t) + C_3sin(t) + ln(|sec(t) + tan(t)|) - tcos(t) + ln(|cos(t)|)sin(t)}$$

Now, solving for C_1, C_2, C_3

$$\begin{cases} y(0) = 2 = C_1 + C_2 \Longrightarrow \boxed{C_1 = 0} \\ y'(0) = 1 = C_3 + 1 - 1 \Longrightarrow \boxed{C_3 = 1} \\ y''(0) = -2 = C_2 \Longrightarrow \boxed{C_2 = 2} \end{cases}$$

The solution to the I.V.P is:

$$\boxed{y = 2cos(t) + sin(t) + ln(|sec(t) + tan(t)|) - tcos(t) + ln(|cos(t)|)sin(t)}$$