Indiana University
Department of Mathematics

Name: Enrique Areyan **Student ID:** 0002876011

Math 343
Midterm

May 28, 2013

You are not allowed to use calculators or any other computational devices. Show all work. *No credit will be given for unsupported answers.*

Either problem 8 OR problem 9 will be graded. Please indicate your choice on the next page. *Only one problem will be graded. If you did not indicate which one to grade, neither will be graded!*

Exam Record

Question 1	5
Question 2	5
Question 3	10
Question 4	5
Question 5	5
Question 6	5
Question 7	10.
Question 8	
Question 9	5
Total	50

Cross which one not to be graded.

1. (*5 points*) Determine an interval where the solution of the given IVP is certain to exist.

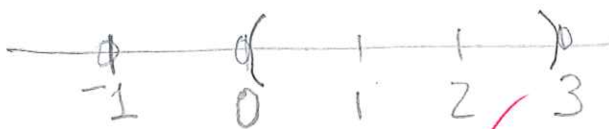
$$(t-3)y' + ln(t+1)y = \frac{1}{t}, \qquad y(1) = 2.$$

Write in standard form: $y' + \frac{\ln(t+1)}{t-3} y = \frac{1}{t(t-3)}$; $y(t=1)=2$

Let $p(t) = \frac{\ln(t+1)}{t-3}$, continuous if: $t+1>0 \Leftrightarrow t>-1$
AND $t-3 \neq 0 \Leftrightarrow t \neq 3$.

$g(t) = \frac{1}{t(t-3)}$, continuos if: $t \neq 0$ AND $t \neq 3$.

$t_0 = 1$.



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By U.E.T, the interval in which the solution is catain to exists is $\boxed{(0,3)}$ Note that $t_0 = 1 \in (0,3)$

2. (*5 points*) Check if the given ODE is exact or not. If it is exact, solve it, if not just find an integrating factor that makes it exact.

$$\underbrace{y}_{M}dx + \underbrace{(2x - ye^y)}_{N}dy = 0$$

$\frac{u'}{u} = \frac{1}{y} \Rightarrow u' = \frac{u}{y}$

the O.D.E is exact if:

$\frac{\partial M}{\partial y} = 1 \neq 2 = \frac{\partial N}{\partial x} \Rightarrow$ it is not exact

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An integrating factor could be: $\frac{u(x)'}{u} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$

$\frac{u'}{u} = \frac{1-2}{2x-ye^y}$; but this is not a pure functio of x, So it does not work.!

try: $\frac{u(y)'}{u} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2-1}{y} = \frac{1}{y}$; this will work!

Solve:

$u'(y) = \frac{u}{y} \Leftarrow u' - \frac{u}{y} = 0$; $I = e^{\int -\frac{1}{y}} = y^{-1}$; $y^{-1}\left[u' - \frac{u}{y} = 0\right]$

back page →

$$+\int \frac{d}{dy}\left[y^{-1}\cdot u\right] = \int 0$$

$$\boxed{u = c \cdot y}$$

this is now the integrating factor since:

$$y\left[y\,dx + (2x - ye^{y})\,dy = 0\right]$$

$$y^2\,dx + 2xy - y^2e^{y} = 0$$

$$\boxed{\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}}$$

3. (*10 points*) Solve the Initial Value Problem and find the domain of validity of the solution.

$$y' + \frac{y}{x} - y^2 = 0, \qquad y(1) = \cancel{0} = 1$$

$y' + \frac{y}{x} = y^2$, Bernoulli equation CASE $n=2$

CHANGE: $u = y^{1-n} = y^{-1} \Rightarrow u' = -1\, y^{-2} y'$

$\Rightarrow \frac{y'}{y^2} = -u'$

$\Leftrightarrow \frac{y'}{y^2} + \frac{1}{xy} = 1 \Leftrightarrow -u' + \frac{u}{x} = 1$

this is a 1st O.D.E. Solve:

(1) standard form: $u' - \frac{u}{x} = -1$.

(2) $u(t) = e^{\int -\frac{1}{x}dx} = e^{\ln(x^{-1})} = x^{-1}$

(3) $x^{-1}\left[u' - \frac{u}{x} = -1\right]$

(4) $\int \frac{d}{dx}\left[x^{-1} u\right] = \int -x^{-1}$

(5) $x^{-1} u = -\ln(x) + C \Rightarrow \boxed{u = -x\ln(x) + Cx}$

Check u sol:

$u = -x\ln(x) + Cx$

$u' = -[1 + \ln(x)] + C$

$-1 - \ln(x) + C - \frac{-x\ln(x) + Cx}{x} =$

$-1 - \ln(x) + C + \cancel{\ln(x)} - \cancel{C} = -1$ ✓

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CHANGE substitution: $u = \frac{1}{y} = -x\ln(x) + Cx \Rightarrow y = \frac{1}{-x\ln(x) + Cx}$

Solve for C: $y(1) = 1 = \frac{1}{-1 \cdot \ln(1) + C \cdot 1} = \frac{1}{C} \Rightarrow \boxed{C = 1}$

The solution is: $\boxed{y(x) = \frac{1}{x(1 - \ln(x))}}$

Domain of validity → BACK PAGE.

this is a non linear, 1^{st} O.D.E.

So we apply appropiate THEOREM:

$$y' = y^2 - \frac{y}{x} = f(x,y)$$

$(\alpha,\beta)\times(\gamma,\delta)$

By U.E.T;

If f, $\frac{\partial f}{\partial y}$ is continuous on an interval

containing $(x_0,y_0)=(1,1)$

then there exists a unique solutio on $(1,1)$.

on an "small box" $x_0+h < x < x_0+h$

Compute $\frac{\partial f}{\partial y} = 2y - \frac{1}{x}$, Also $f = y^2 - \frac{y}{x}$;

these functions are continuous everywhere except

when $x=0$. Hence, there is a solution on

small box

$1-h < x < 1+h$

Now, looking at the actual solution:

$y(x) = \frac{1}{x(1-\ln(x))}$, we can conclude that the solution is valid only if $x>0$.

4. (*10 points*) Given $y_1(t) = t^{-1}$ a solution of the differential equation

$$2t^2y'' + ty' - 3y = 0.$$

Use the Method of Reduction of Order to find $y_2(t)$.

$\boxed{y_2(t) = v \cdot t^{-1}}$

$\boxed{y_2'(t) = v' \cdot t^{-1} - v \cdot t^{-2}}$

$y_2''(t) = v'' t^{-1} - v' t^{-2} - [v \cdot -2t^{-3} + v' \cdot t^{-2}]$

$\boxed{y_2''(t) = v'' \cdot t^{-1} - 2v' \cdot t^{-2} + 2v \cdot t^{-3}}$

y_2 satisfies the equation:

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$2t^2(v'' t^{-1} - 2v' t^{-2} + 2v t^{-3}) + t(v' t^{-1} - v t^{-2}) - 3(v \cdot t^{-1}] = 0$

$2t v'' - 4v' + 4vt^{-1} + v' - vt^{-1} - 3v \cdot t^{-1} = 0$

$2t v'' + v'[-4+1] + v\{4t^{-1} - t^{-1} - 3t^{-1}] = 0$

$2t^2 v'' - 3v' = 0$. CHANGE: $w = v' \Rightarrow w' = v''$

$\Leftrightarrow$ $2t^2 w' - 3w = 0$ this is a 1st O.D.E. linear:

(1) $w' - \frac{3}{2t^2} w = 0$; (2) $u(t) = e^{\int -\frac{3}{2}t^2} = e^{+\frac{3}{2}\frac{1}{t}} = e^{\frac{3}{2t}}$

(3) $e^{\frac{3}{2t}}[w' - \frac{3}{2t^2} w = 0]$ (4) $\int \frac{d}{dt}[e^{\frac{3}{2t}} \cdot w] = \int 0$

(5) $e^{\frac{3}{2t}} \cdot w = C \Rightarrow \boxed{w = e^{-\frac{3}{2t}}}$; change back to v.

$w = v' = e^{-\frac{3}{2t}} \Rightarrow v = \int e^{-\frac{3}{2t}} dt \Rightarrow \boxed{v = -\frac{2}{3} e^{-\frac{3}{2t}}}$

So, our second solution is $y_2(t) = v \cdot t^{-1} \Rightarrow \boxed{y_2(t) = -\frac{2}{3t} \cdot e^{-\frac{3}{2t}}}$ //

5. (*5 points*) Solve the following Initial Value Problem.

$(\lambda \pm \mu i)$

$$y'' + 4y = 0, \qquad y(0) = 0, \quad y'(0) = 1$$

CHARACTERISTIC EQUATION: $r^2 + 4 = 0 \Rightarrow r = \sqrt{-4} \Rightarrow r = \pm 2i$

General solution CASE complex roots:

$$y(t) = C_1 e^{0t}\cos(2t) + C_2 e^{0t}\sin(2t)$$

$$\boxed{y(t) = C_1\cos(2t) + C_2\sin(2t)}$$

Solve for C1, C2:

$$y(0) = \boxed{C_1 = 0}$$

$$y'(0) = 2C_2\cos(0) = 2C_2 = 1 \Rightarrow \boxed{C_2 = \frac{1}{2}}$$

Solution to the I.V.P. IS:

$$\boxed{y(t) = \frac{1}{2}\sin(2t)}$$

CHECK solution:
$y = \frac{1}{2}\sin(2t)$, $y' = \cos(2t)$
$y'' = -2\sin(2t)$; $-2\sin(2t) + 2\sin(2t)$
$= 0$
$\Rightarrow$ solution works!

6. (*5 points*) Use Euler's Method with $h = 0.05$ to find $y(0.1)$ of the given IVP.

$$y' = t - 2y, \qquad y(0) = 1$$

Euler's Method $y_{n+1} = y_n + hf(t_n, y_n)$; $t_n = t_0 + nh$

Hence: Given that $f(t,y) = t - 2y$ AND $t_0 = 0$; $y_0 = 1$.

We compute:

$$y_1 = y_0 + hf(t_0, y_0) = 1 + 0.05(0 - 2(1)) = 1 - 0.10$$

$$\Rightarrow \boxed{y_1 = 0.9} \text{ which means } \boxed{y(0.05) = 0.9}$$

Finally:

$$y_2 = y_1 + hf(t_1, y_1) = 0.9 + 0.05(0.05 - 2(0.9))$$
$$= 0.9 + 0.05(0.05 - 1.8)$$
$$= 0.9 + 0.05(-1.75)$$
$$= 0.9 - 0.0875$$
$$\boxed{= 0.8125}$$

Hence

$$\boxed{y(0.1) = 0.8125}$$

7. (*10 points*) Solve the initial value problem:

$$y'' - 2y' + y = 3e^t + \cos(t), \qquad y(0) = 0, \quad y'(0) = 1.$$

General Solution given by:

$y_g = y_h + y_p$, where:

y_h: $y'' - 2y' + y = 0$ CHARACTERISTIC EQUATION $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

General solution, CASE REPEATED ROOTS:

$\boxed{y_h = C_1 e^t + C_2 t e^t}$

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y_p: $3e^t$ is Linearly dependent with y_h. Hence, use:

$y_p = At^2e^t + B\sin(t) + C\cos(t)$. then:

$\boxed{y_p' = 2Ate^t + At^2e^t + B\cos(t) - C\sin(t)}$

$y_p'' = 2Ae^t + 2Ate^t + 2Ate^t + At^2e^t - B\sin(t) - C\cos(t)$

y_p satisfies the eq

$y_p'' - 2y_p' + y_p = 3e^t + \cos(t)$

$2Ae^t + 4Ate^t + At^2e^t - B\sin(t) - C\cos(t) - 4Ate^t - 2At^2e^t - 2B\cos(t) + 2C\sin(t)$
$+ At^2e^t + B\sin(t) + C\cos(t) = 3e^t + \cos(t)$

$$\begin{cases} e^t(2A + 4At + At^2 - 4At - 2At^2 + At^2) = 3e^t \\ \sin(t)(-B + 2C + B) + \cos(t)(-C - 2B + C) = \cos(t) \end{cases}$$

$$\begin{cases} 2A = 3 \Rightarrow \boxed{A = \frac{3}{2}} \\ 2C = 0 \Rightarrow \boxed{C = 0} \\ -2B = 1 \Rightarrow \boxed{B = -1/2} \end{cases}$$

Solution: back page: $\longrightarrow$

$y_g = y_h + y_p$

$$\boxed{y_g = C_1 e^t + C_2 t e^t + \frac{3}{2} t^2 e^t - \frac{1}{2}\sin(t)}$$

Solving for C_1, C_2:

$y(0) = \boxed{0 = C_1}$

$y'(0) = 1 = C_2 - \frac{1}{2}\cos(0) = C_2 - \frac{1}{2} \Rightarrow \boxed{C_2 = \frac{3}{2}}$

the solution to the I.V.P is:

$$\boxed{y = \frac{3}{2} t e^t + \frac{3}{2} t^2 e^t - \frac{1}{2}\sin(t)}$$

8. *(10 points)*

a. *(5 pts)* If the Wronskian W of f and g is t^2e^t, and if $f(t) = t$, find $g(t)$.

b. *(5 pts)* Solve the differential equation

$$5y + x - (y - 5x)y' = 0.$$

1h → 60 min

9. (*10 points*) A tank originally contains 50 gal of fresh water. Water containing 3/2 lb of salt per gallon in entering the tank at rate 2gal/min and the mixture is allowed to leave the tank at a rate of 30gal/hr. Find the amount of salt in the tank after 10 min.

Let $Q(t)$ = amount, in lb of salt at minute t.

the model of this situation is:

$$\begin{cases} \frac{dQ}{dt} = \text{rate in} - \text{rate out}, \\ Q(0) = 0 \end{cases}$$

NOTE THAT $30\frac{gal}{h} = \frac{30}{60} = \frac{3}{6} = \frac{1}{2}\frac{gal}{min}$

$$\Rightarrow \begin{cases} \frac{lb}{min}\ \frac{dQ}{dt} = 2\frac{gal}{min} \cdot \frac{3}{2}\frac{lb}{gal} - \frac{1}{2}\frac{gal}{min} \cdot \frac{Q(t)}{V(t)}\frac{lb}{gal} \\ Q(0) = 0 \end{cases}$$

where, $V(t)$ is given by: $\frac{dV}{dt} = 2 - \frac{1}{2} = \frac{3}{2}$ 5

$\Rightarrow V(t) = \frac{3}{2}t + C$, but $V(0) = 50 = C$

hence $\boxed{V(t) = \frac{3}{2}t + 50}$. the final model is:

$$\begin{cases} \frac{dQ}{dt} = 3 - \frac{Q(t)}{t+100} \\ Q(0) = 0. \end{cases}$$ X

this is a linear. 1st O.D.E

Solve by integrating factor

$Q' + \frac{1}{t+100}Q = 3$ X $u(t) = e^{\int\frac{1}{t+100}} = e^{\ln(t+100)} = t+100$

$[t+100]\left[Q' + \frac{1}{t+100}Q = 3\right] \Rightarrow \frac{d}{dt}\left[(t+100).Q\right] = 3t + 300$

BACK page. →

$$(t+100)\cdot Q = \int 3t + 300 = \frac{3}{2}t^2 + 300t + C$$

$$\Rightarrow \boxed{Q = \frac{\frac{3}{2}t^2 + 300t + C}{t+100}},$$

Solving for C:

$$Q(0) = \frac{C}{100} = 0 \Rightarrow C = 0.$$

Our model for this I.V.P is

$$\boxed{Q(t) = \frac{\frac{3}{2}t^2 + 300t}{t+100}}$$

After 10 minutes.

$$Q(10) = \frac{\frac{3}{2}\cdot 100 + 3000}{110} = \frac{150+3000}{110} = \frac{315\cancel{0}}{11\cancel{0}}$$

$$= \boxed{\frac{315}{11}} \text{ lbs of salt}$$