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(1) Let $C \in M_n(F)$ and suppose $v^t C w = v^t w$ for all $v, w \in F^n$. Prove $C = I_n$.

Pf: Since the condition hold for all elements of F^n, in particular it holds for e_i where $e_i = (0\ 0 \cdots 1 \cdots 0)^t$ (ith position). therefore, apply the condition for e_i and e_j with $1 \le i, j \le n$: then $v^t w = e_i^t e_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

First note that if $v^t = e_i$ and $w = e_j$

then, $v^t C w = e_i(C e_j) = e_i(C_j)$, where C_j represents the j^{th} column of C
$= C_{ij}$, where C_{ij} represents the (i,j) entry of C
$= e_i^t e_j$, by hypothesis

therefore, $C_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise,} \end{cases}$ that is, $C = I_n$. +10

(3) Determine whether the following matrix over F_{11} can be diagonilized:

$$A = \begin{pmatrix} 5 & 5 & -1 \\ -2 & 4 & -5 \\ 2 & -3 & 6 \end{pmatrix}.$$

Solution: Let us find its eigenvalues. For that purpose let us compute the $ch_A(\lambda)$

$\det(A - \lambda I) = 0 \Rightarrow \det\left[\begin{pmatrix} 5 & 5 & -1 \\ -2 & 4 & -5 \\ 2 & -3 & 6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right] = \det\begin{bmatrix} 5-\lambda & 5 & -1 \\ -2 & 4-\lambda & -5 \\ 2 & -3 & 6-\lambda \end{bmatrix}$

$= (5-\lambda)\begin{bmatrix} 4-\lambda & -5 \\ -3 & 6-\lambda \end{bmatrix} - 5\begin{bmatrix} -2 & -5 \\ 2 & 6-\lambda \end{bmatrix} + (-1)\begin{bmatrix} -2 & 4-\lambda \\ 2 & -3 \end{bmatrix}$

$= [(5-\lambda)[(4-\lambda)(6-\lambda) - 15]] - 5[(-2)(6-\lambda) + 10] - 1[6 - [(2)(4-\lambda)]]$

$= [(5-\lambda)[(4-\lambda)(6-\lambda) - 15]] - 5[-12 + 2\lambda + 10] - 1[6 - 8 + 2\lambda]$

$= [(5-\lambda)[(4-\lambda)(6-\lambda) - 15]] - 5[2\lambda - 2] - 1[2\lambda - 2]$

$= [(5-\lambda)[(4-\lambda)(6-\lambda) - 15]] - 6[2\lambda - 2]$

$= [(5-\lambda)[(4-\lambda)(6-\lambda) - 15]] - 12\lambda + 12$

$= [(5-\lambda)[(24 - 10\lambda + \lambda^2) - 15]] - 12\lambda + 12$

$= [(5-\lambda)(\lambda^2 - 10\lambda + 9)] - 12\lambda + 12$ 8

$= [5\lambda^2 - 50\lambda + 45 - \lambda^3 + 10\lambda^2 - 9\lambda] - 12\lambda + 12$

$= -\lambda^3 + 15\lambda^2 - 71\lambda + 57$

Check! A is indeed diagonalize

$\Rightarrow ch_A(\lambda) = -\lambda^3 + 15\lambda^2 - 71\lambda + 57$, in the field F_{11} we have

$\boxed{ch_A(\lambda) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2}$

We can easily see that $\lambda = 1$ is a root, so we can factor ch_A as:

$ch_A(\lambda) = (\lambda - 1)(-\lambda^2 + 3\lambda - 2)$,

$\boxed{ch_A(\lambda) = (\lambda-1)(\lambda-1)(\lambda-2)}$ continue ⊛

⊛. So we have found that the eigenvalues of A over F_{11} are $\lambda = 1$ and $\lambda = 2$.

we better check the dimension of the eigenspace $V_1 = \{v \in V : Av = v\}$

$\begin{bmatrix} 5 & 5 & -1 \\ -2 & 4 & -5 \\ 2 & -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{cases} 5x_1 + 5x_2 - x_3 = x_1 \\ -2x_1 + 4x_2 - 5x_3 = x_2 \\ 2x_1 - 3x_2 + 6x_3 = x_3 \end{cases}$

$\begin{cases} 4x_1 + 5x_2 - x_3 = 0 \\ -2x_1 + 3x_2 - 5x_3 = 0 \\ 2x_1 - 3x_2 + 5x_3 = 0 \end{cases}$ $\Rightarrow$ this is a multiple of the last eq, it provides no information

$\begin{cases} 4x_1 + 5x_2 - x_3 = 0 \\ 2x_1 - 3x_2 + 5x_3 = 0 \end{cases}$ $\stackrel{(x2)}{\Leftrightarrow}$ $\begin{cases} 4x_1 + 5x_2 - x_3 = 0 \\ -4x_1 + 6x_2 - 10x_3 = 0 \end{cases}$ $0x_1 + 11x_2 - 11x_3 = 0 \Leftrightarrow x_2 - x_3$

$\Leftrightarrow \boxed{x_2 = x_3}$

Replacing in first eq: $4x_1 + 5x_2 - x_2 = 4x_1 + 4x_2 = 0$
$\Leftrightarrow x_1 + x_2 = 0 \Leftrightarrow \boxed{x_1 = -x_2 = -x_3}$. Hence:

$V_1 = \left\langle \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\rangle$. therefore,

$\dim(V_1) = 1$, not enough for the matrix to be diagonalizable over F_{11}.

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(2) For any field F we can define the orthogonal group $O_n(F)$ as follow
$O_n(F) = \{A \in M_n(F) \mid A^t A = I_n\}$. It is easy to see that $O_n(F) \leq Gl_n(F)$.

(a) Prove that $O_2(F) = \left\{ \begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix} \mid a,b \in F \text{ and } a^2+b^2=1 \right\}$.

Pf: By double contention:

($\supseteq$) Let $A = \begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix}$, where $a,b \in F$ and $a^2+b^2=1$. then,

$$A^t A = \begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix}^t \begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix} = \begin{pmatrix} a & \pm b \\ b & \mp a \end{pmatrix}\begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix} = \begin{pmatrix} a^2+(\pm b)^2 & ab+(\pm b)(\mp a) \\ ab+(\pm b)(\mp a) & b^2+(\mp a)^2 \end{pmatrix}$$

$$= \begin{pmatrix} a^2+b^2 & ab-ab \\ ab-ab & a^2+b^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2. \text{ Hence, } A \in O_2(F).$$

($\subseteq$) Let $A \in O_2(F)$. By definition, $A^t A = I_n$. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a,b,c,d \in F$
then, $A^t A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^t \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{cases} a^2+c^2 = 1 \\ ab+cd = 0 \\ b^2+d^2 = 1 \end{cases}$ But recall that $O_2(F) \leq Gl_n(F)$, therefore,
$ad - bc \neq 0 \iff ad \neq bc$.

which together with $ab+cd=0 \iff ab = -cd$. From this we can conclude that
Imply that $c = \pm b$ and $d = \mp a$.
$a^2+b^2=1$, obtaining the result, i.e.,

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ \pm b & \mp a \end{bmatrix}$, Hence $A = \left\{ \begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix} \mid a,b \in F \text{ and } a^2+b^2=1 \right\}$.

Together, ($\subseteq$) & ($\supseteq$) $\Rightarrow O_2(F) = \left\{ \begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix} \mid a,b \in F \text{ and } a^2+b^2=1 \right\}$.

(b) Find $|O_2(F_7)|$ and $|O_2(F_{11})|$. Identify the group $O_2(F_7)$.

Solution: $|O_2(F_7)| = \left| \left\{ \begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix} \mid a,b \in F_7 \text{ and } a^2+b^2=1 \right\} \right|$, by part (a).
So first, let us find $a,b \in F_7$ s.t. $a^2+b^2=1$. Considering $F_7 = \{-3,-2,-1,0,1,2,$
we need only to check $a,b \in \{0,1,2\}$ (squares of F_7). the following tabl
summarizes the operation a^2+b^2 on $\{0,1,2\}$ over F_7:

a\b	0	1	2
0	0	1	4
1	1	2	5
2	4	5	1

(Again, entries of this table are a^2+b^2 m
We can see that the only numbers th
work (in terms of $O_2(F_7)$, i.e, $a^2+b^2=1$)

$(a,b) = (0,1), (1,0), (0,-1), (-1,0), (2,2), (2,-2), (-2,2), (-2,-2)$.
Each of these pairs corresponds to two matrices in $O_2(F_7)$, therefor

$|O_2(F_7)| = 2 \cdot 8 = \boxed{16}$

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Note that $O_2(F_7)$ is not abelian: Consider $\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&-1\\-1&0\end{pmatrix} \in O_2(F_7)$

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-1\\-1&0\end{pmatrix} = \begin{pmatrix}0&-1\\1&0\end{pmatrix} \neq \begin{pmatrix}0&-1\\-1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$O_2(F_7)$ is isomorphic to D_{16}. $\boxed{B = \Theta(p) \quad R = \Theta(r) \Rightarrow rpr^{-1} = p^{-1}}$

Using the same reasoning as before, we can compute $|O_2(F_{11})|$. We will need to look at $a, b \in \{0,1,3,4,5,9\}$, s.t. $a^2+b^2 = 1$.

a\b	0	1	3	4	5	9
0	0	1	9	5	3	4
1	1	2	10	6	4	5
3	9	10	7	3	1	2
4	5	6	3	10	8	9
5	3	4	1	8	6	7
9	4	5	2	9	7	8

Possible values for a and b are: $(a,b) = (0,1), (1,0), (0,-1), (-1,0), (3,5), (5,3), (-3,5), (5,-3), (3,-5), (-5,3), (-3,-5), (-5,-3)$.

Again, each of these pairs correspond to two matrices, for example
$(5,-3) \mapsto \left\{\begin{pmatrix}5&-3\\-3&-5\end{pmatrix}, \begin{pmatrix}5&-3\\3&5\end{pmatrix}\right\}$, and so on with all others.

Therefore, $|O_2(F_{11})| = 2 \cdot 12 = \boxed{24}$

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(4) (a) Let A be an $m\times m$ matrix over F and let B be an $n\times n$ matrix over F. Show that if C is any $m\times n$ matrix over F then:

$$\det\begin{pmatrix}A & C\\ O_{n\times m} & B\end{pmatrix} = \det(A)\det(B)$$

Pf: Note that we can always factor the block matrix $\begin{pmatrix}A & C\\ O & B\end{pmatrix}$ as follows:

$$\begin{pmatrix}A_{m\times m} & C_{m\times n}\\ O_{n\times m} & B_{n\times n}\end{pmatrix} = \begin{pmatrix}A_{m\times m} & AP+QB\\ O_{n\times m} & B_{n\times n}\end{pmatrix} = \begin{pmatrix}A_{m\times m} & Q_{m\times n}\\ O_{n\times m} & I_n\end{pmatrix}\begin{pmatrix}I_m & P_{m\times n}\\ O_{n\times m} & B_{n\times n}\end{pmatrix},$$

where $C_{m\times n} = \underbrace{\overbrace{A}^{m\times m}\overbrace{P}^{m\times n}}_{m\times n} + \underbrace{\overbrace{Q}^{m\times n}\overbrace{B}^{n\times n}}_{m\times n}$. We can always find such P, Q by solving the system $AP+QB = C$, where we will have more variables than equations (so we are sure to get a solution).

Hence, we only need to show that $\det\begin{pmatrix}A & Q\\ 0 & I\end{pmatrix} = \det(I)$ ... $\det\begin{pmatrix}I & P\\ 0 & B\end{pmatrix} = \det(B)$ follows ... prove what we want (Note that $\det\begin{pmatrix}I & P\\ 0 & B\end{pmatrix}$ ... from $\det\begin{pmatrix}A & Q\\ 0 & I\end{pmatrix}$ by using a very similar argument).

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Let us prove: $\det\begin{pmatrix} A_{m\times m} & B_{m\times n} \\ O_{n\times m} & I_n \end{pmatrix} = \det(A)$, by induction on n.

BASE CASE: If $n=1$:

$\det\begin{pmatrix} A & \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix} \\ O & 1 \end{pmatrix} = (-1)^{2(m+1)}\det(A) + $ (a bunch of zeros from last row) $= \det(A)$. BASE CASE HOLDS.

Inductive STEP: Suppose that the result hold for all $1\leq k\leq n-1$. We want to show that it holds for n.

By induction hypothesis

$$\det\begin{pmatrix} A_{m\times m} & B_{m\times n} \\ O_{n\times m} & I_n \end{pmatrix} = \det\begin{pmatrix} A & B' & \begin{matrix} b_{n1} \\ \vdots \\ b_{nn} \end{matrix} \\ O & I_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0\ 0 \cdots & 0 & 1 \end{pmatrix} = (-1)^{2(n+m)}\det\begin{pmatrix} A & B'_{m\times(n-1)} \\ O & I_{n-1} \end{pmatrix} = \det(A) \quad \square$$

Now apply this to:

$$\det\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det\begin{pmatrix} A & AP+QB \\ 0 & B \end{pmatrix} = \det\left(\begin{pmatrix} A & Q \\ 0 & I \end{pmatrix}\begin{pmatrix} I & P \\ 0 & B \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} A & Q \\ 0 & I \end{pmatrix}\det\begin{pmatrix} I & P \\ 0 & B \end{pmatrix}$$

+10 $= \det(A)\det(B)$

(b) Let V be an F-vector space and let $T: V\to V$ be a linear transformation. Let W be a T-invariant subspace. Let $\bar{T}: V/W \to V/W$ be the induced linear transformation. Prove that $ch_{T|_W}(x)\, ch_{\bar{T}}(x) = ch_T(x)$.

Pf: Since W is T-invariant, we know that there exists a basis of B such that: $M_B(T) = \left(\begin{array}{c|c} M_{B_1}(T|_W) & * \\ \hline O & M_{\bar{B_2}}(\bar{T}) \end{array}\right)$, where

$B = (\underbrace{w_1,\dots,w_m}_{B_1}, \underbrace{w_{m+1},\dots,w_n}_{B_2})$, where B_1 is a basis of W and you complete that basis to get a basis for (we proved this in class). We also know that $ch_T(x) = ch_{M_B(T)}(x)$ for any choice of a basis B. So, choosing the above basis:

$$\boxed{ch_T(x)} = \det\left(\left[\begin{array}{c|c} M_{B_1}(T|_W) & * \\ \hline O_{(n-m)\times m} & M_{\bar{B_2}}(\bar{T})_{(n-m)\times(n-m)} \end{array}\right]_{n\times n} - x\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}\right) = \det\left(\left[\begin{array}{c|c} M_{B_1}(T|_W)-\lambda I_m & * \\ \hline 0 & M_{\bar{B_2}}(\bar{T})-\lambda I_{(n-m)} \end{array}\right]\right)$$

(by previous part) $= \det(M_{B_1}(T|_W)-\lambda I_m)\cdot\det(M_{\bar{B_2}}(\bar{T})-\lambda I_{(n-m)}) = \boxed{ch_{T|_W}(x)\cdot ch_{\bar{T}}(x)}$.

the result follows by definition of characteristic polynomial.

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(5) Prove that if B and C are orthonormal bases for $\mathbb{R}^n$, then the change of basis matrix from B to C is orthogonal.

Pf: Let $B=(b_1,\dots,b_n)$ and $C=(c_1,\dots,c_n)$ be orthonormal bases for $\mathbb{R}^n$.
By definition, $b_i\cdot b_j=\begin{cases}0 & \text{if } i\neq j\\ 1 & \text{if } i=j\end{cases}$, Likewise $c_i\cdot c_j=\begin{cases}0 & \text{if } i\neq j\\ 1 & \text{if } i=j\end{cases}$.
Consider the change of basis matrix from the basis B to the basis C:

$$\underset{B\to C}{P}=\left[[b_1]_C \vdots [b_2]_C \vdots \cdots \vdots [b_n]_C\right]$$

We want to show that P is orthogonal, that is: $P^TP=I_n$. Equivalenty: $[b_i]_C\cdot[b_j]_C \overset{?}{=} \begin{cases}0 & \text{if } i\neq j\\ 1 & \text{otherwise}\end{cases}$. Pf: consider:

claim: Since C is a orthonormal basis, $[b_i]_C=\sum_{k=1}^n \langle b_i,c_k\rangle c_k$. This is true because: $[b_i]_C=\alpha_1c_1+\alpha_2c_2+\dots+\alpha_nc_n$. We want to show $\alpha_j=\langle b_i,c_j\rangle$, $j=1,2,\dots,n$. But then, $\langle b_i,c_j\rangle=\langle\sum_{i=1}^n\alpha_jc_i,c_j\rangle=\alpha_1\langle c_1,c_j\rangle+\alpha_2\langle c_2,c_j\rangle+\dots+\alpha_n\langle c_n,c_j\rangle$ $=\alpha_j\langle c_j,c_j\rangle=\alpha_i$ because C is an orthonormal basis. □

$\langle[b_i]_C,[b_j]_C\rangle=\langle\sum_{k=1}^n\langle b_i,c_k\rangle c_k,\sum_{k=1}^n\langle b_j,c_k\rangle c_k\rangle=$ (by expanding)
$=\sum_{k=1}^n\langle b_i,c_k\rangle\cdot\langle b_j,c_k\rangle=\begin{cases}0 & \text{if } i=j\\ 1 & \text{otherwise},\end{cases}$ which shows the result.

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(6) Let W be a subspace of $\mathbb{R}^n$ and let $(w_1,\dots,w_k)$ be a basis for W.
Define vectors $\tilde{w}_1,\tilde{w}_2,\dots,\tilde{w}_k$ as follows:
$\tilde{w}_1=w_1$ and for $0<i<k$, $\tilde{w}_{i+1}=w_{i+1}-\sum_{j=1}^i\frac{(w_{i+1},\tilde{w}_j)}{(\tilde{w}_j,\tilde{w}_j)}\tilde{w}_j$.
Prove that $\tilde{w}_1,\dots,\tilde{w}_k$ is an orthogonal basis for W.

Pf: By induction on the dimension of W. Consider the statement
$S(k)$: $\{\tilde{w}_1,\dots,\tilde{w}_k\}$ is an orthogonal basis for W. So $\dim(W)=k$.
Of course here $0<k\leq n$, since W is a subspace of $\mathbb{R}^n$.

BASE CASE: $S(1)$: $\{\tilde{w}_1\}$ is trivially a orthogonal basis for W.
Inductive STEP: Suppose that $S(m)$ is true for some m, that is: $\{\tilde{w}_1,\dots,\tilde{w}_m\}$ is a orthogonal basis for W. where $\dim(W)=k$. We want to show that $S(m+1)$ is true. So the goal is to show that $\{\tilde{w}_1,\dots,\tilde{w}_m,\tilde{w}_{m+1}\}$ is a basis for the space W', where $\dim(W$
Note that if $m=n$, then there is nothing to prove so assume that $m+1\leq n$. Let us first show that $\{\tilde{w}_1,\dots,\tilde{w}_{m+1}\}$ is an orthogonal se

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By inductive hypothesis, $\{\tilde{w}_1, \ldots, \tilde{w}_m\}$ is an orthogonal set. Hence, we need to show that if we add $\tilde{w}_{m+1}$ the set remains orthogonal. But that amounts to show that this new vector is orthogonal with the old vectors. that is: $\langle \tilde{w}_i, \tilde{w}_{m+1} \rangle = 0$ for $0 < i \leq m$. this is true because:

$$\langle \tilde{w}_i, \tilde{w}_{m+1} \rangle = \langle \tilde{w}_i, w_{m+1} - \sum_{j=1}^{m} \frac{\langle w_{m+1}, \tilde{w}_j \rangle}{\langle \tilde{w}_j, \tilde{w}_j \rangle} \tilde{w}_j \rangle$$

By properties of dot product: $\langle a, b+c \rangle = \langle a, b \rangle + \langle b, c \rangle$

$$= \langle \tilde{w}_i, w_{m+1} \rangle - \left[\langle \tilde{w}_i, \sum_{j=1}^{m} \frac{\langle w_{m+1}, \tilde{w}_j \rangle}{\langle \tilde{w}_j, \tilde{w}_j \rangle} \tilde{w}_j \rangle \right]$$

$$= \langle \tilde{w}_i, w_{m+1} \rangle - \left[\langle \tilde{w}_i, \frac{\langle w_{m+1}, \tilde{w}_1 \rangle}{\langle \tilde{w}_1, \tilde{w}_1 \rangle} \tilde{w}_1 \rangle + \cdots + \langle \tilde{w}_i, \frac{\langle w_{m+1}, \tilde{w}_m \rangle}{\langle \tilde{w}_m, \tilde{w}_m \rangle} \tilde{w}_m \rangle \right.$$

Taking out scalars.

$$= \langle \tilde{w}_i, w_{m+1} \rangle - \frac{\langle w_{m+1}, \tilde{w}_1 \rangle}{\langle \tilde{w}_1, \tilde{w}_1 \rangle} \langle \tilde{w}_i, \tilde{w}_1 \rangle - \cdots - \frac{\langle w_{m+1}, \tilde{w}_m \rangle}{\langle \tilde{w}_m, \tilde{w}_m \rangle} \langle \tilde{w}_i, \tilde{w}_m \rangle$$

$$= \langle \tilde{w}_i, w_{m+1} \rangle - \frac{\langle w_{m+1}, \tilde{w}_i \rangle}{\langle \tilde{w}_i, \tilde{w}_i \rangle} \langle \tilde{w}_i, \tilde{w}_i \rangle$$

, by inductive hypothesis, $\langle \tilde{w}_i, \tilde{w}_j \rangle = 0$ if $i \neq j$, for $0 < i \leq m$.

Cancelling common factor

$$= \langle \tilde{w}_i, w_{m+1} \rangle - \langle w_{m+1}, \tilde{w}_i \rangle$$

+10

since the dot product is symmetric

$$= \langle \tilde{w}_i, w_{m+1} \rangle - \langle \tilde{w}_i, w_{m+1} \rangle$$

$= \boxed{0}$ Hence, $\{\tilde{w}_1, \ldots, \tilde{w}_{m+1}\}$ is an orthogonal set.

Now we need to prove that $\text{span}\{w_1, \ldots, w_{m+1}\} = \text{span}\{\tilde{w}_1, \ldots, \tilde{w}_{m+1}\}$

By inductive hypothesis $\text{span}\{w_1, \ldots, w_m\} = \text{span}\{\tilde{w}_1, \ldots, \tilde{w}_m\}$. Hence, it suffices to show that $w_{m+1} \in \text{span}\{\tilde{w}_1, \ldots, \tilde{w}_{m+1}\}$ and $\tilde{w}_{m+1} \in \text{span}\{w_1, \ldots, w_{m+1}\}$. the later follows from the recursive definition of $\tilde{w}$, i.e., you produce $\tilde{w}_i$ by taking linear combination with appropiate coefficients of w_j ; $0 \leq j < i$.

Likewise, this fact shows that $w_{m+1} \in \text{span}\{\tilde{w}_1, \ldots, \tilde{w}_{m+1}\}$.

So this means that we have gained nothing new in Terms of what is being spanned by the set $\{w_1, \ldots, w_{m+1}\}$ in relation to $\{\tilde{w}_1, \ldots, \tilde{w}_{m+1}\}$.

Finally, since we assumed that $\{w_1, \ldots, w_{m+1}\}$ is a basis of W, by our previous reasoning (about span), we can conclude that $\{\tilde{w}_1, \ldots, \tilde{w}_{m+1}\}$ is linearly independent and hence, a orthogonal basis of W

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(7)(a) Prove that if n is an integer $n \geq 3$, then D_n, the dihedral group of order 2n, is isomorphic to a subgroup of $O_2(\mathbb{R})$.

Pf: We proved in 2 (a) that:

$$O_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix} \middle| a, b \in \mathbb{R} \quad a^2+b^2=1 \right\}.$$

We want to find $f: D_n \to O_2(\mathbb{R})$ s.t f is an isomorphism, provided that $n \geq 3$ is an integer.

We know that the dihedral group of order n is generated by two elements: $\begin{cases} \rho = \text{rotation by } \frac{2\pi}{n} \\ r = \text{reflection across y-axis.} \end{cases}$

$D_n = \langle \rho, r \rangle$.

Now, the group $O_2(\mathbb{R})$ is the group of all rotations by an angle θ in $\mathbb{R}^2$ togheter with reflections (it suffices to consider only one rotation - say across the x or y - axis). Clearly, if we restrict ourselves to elements in $O_2(\mathbb{R})$ that rotate by $\frac{2\pi}{n}$ and reflect across the y-axis, then we will get a subgroup with exactly 2n elements. In fact this subgroup will be isomorphic to Dn $(n \geq 3)$, which can be explained by the geometric action of the group in $\mathbb{R}^2$, that is: rigid rotations and reflection. In particular all of these preserve lengths and map vertices to vertices which is exactly what we wanted.

In short, to talk about dihedral groups D_n; $(n \geq 3)$ is really the same as talking about all rigid motions of the n-gon, which is for sure a subgroup of all the rigid motions of $\mathbb{R}^2$, that is, $O_2(\mathbb{R})$.

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7 (b) Prove that:

$O_2(\mathbb{R}) = \{$ group of symmetries of the unit circle $\}$
$= \{ f \mid f$ is a 1-1, onto function from the unit circle to itself that preserve distance $\}$.

Pf: We may use the characterization of $O_2(\mathbb{R})$ as proved in 2(a)
We know that $O_2(\mathbb{R}) = \{ \begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix} \mid a,b \in \mathbb{R}$ and $a^2+b^2=1 \}$.
So it suffices to show that:

$\{ f \mid f$ is 1-1, onto, unit circle to itself (preserves distance) $\} = \{ \begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix} \mid a,b \in \mathbb{R}$ and $a^2+b^2=1 \}$

($\supseteq$) We want to show that $\begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix}$ (viewed as a function $\mathbb{R}^2 \to \mathbb{R}^2$) is (i) from unit circle to itself. (ii) 1-1, (iii) onto, (iv) distance preserving

(i) Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be an arbitrary point in the unit circle. ($x^2+y^2=1$). Then
$\begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ \pm bx + (\mp ay) \end{bmatrix}$; this is a point on the unit circle because

$$(ax+by)^2 + (\pm bx + (\mp ay))^2 = a^2x^2 + b^2y^2 + 2abxy + b^2x^2 + a^2y^2 \pm 2(\pm bx)(\mp ay)$$
$$= x^2(a^2+b^2) + y^2(a^2+b^2) + 2abxy - 2abxy$$
$$= x^2+y^2 = 1.$$

(ii) 1-1: Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in C$ be two different points on the unit circle. Then, we can write $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \sin\varphi \\ \cos\varphi \end{bmatrix}$, where $\theta \neq \varphi$.

$$\begin{bmatrix} a & b \\ \pm b & \mp a \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} a\cos\theta + b\sin\theta \\ \pm b\cos\theta + \mp a\sin\theta \end{bmatrix} \neq \begin{bmatrix} a\cos\varphi + b\sin\varphi \\ \pm b\cos\varphi + \mp a\sin\varphi \end{bmatrix} = \begin{bmatrix} a & b \\ \pm b & \mp a \end{bmatrix} \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix},$$

therefore, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, so $\begin{bmatrix} a & b \\ \pm b & \mp a \end{bmatrix}$ is 1-1.

(iii) onto: we want to show that $\forall \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in C$ (unit circle) $\exists \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in C$: $\begin{bmatrix} a & b \\ \pm b & \mp a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ ...
But note that $\det \begin{bmatrix} a & b \\ \pm b & \mp a \end{bmatrix} \neq 0 \Rightarrow \begin{bmatrix} a & b \\ \pm b & \mp a \end{bmatrix}$ is invertible so we can always find such a point. In fact this argument works for (ii) as well.

(iv) that $\begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix}$ is distance preserving follows from definition since it belongs to $O_2(\mathbb{R})$.

($\subseteq$) A function with all said characteristic can be viewed as a matrix (of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ \pm\sin\theta & \mp\cos\theta \end{pmatrix}$), so the above argument is sufficient.