M403- HW 11.- Fall 2013- Enrique Areyan   60/60   1

~~6.3~~ (3.1) Verify the rules (6.3.3)

(i) w.t.s $\rho_\theta t_v \stackrel{?}{=} t_{v'}\rho_\theta$, where $v' = \rho_\theta(v)$; Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

on the one hand:

$\rho_\theta t_v(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1+v_1 \\ x_2+v_2 \end{bmatrix}$

$= \begin{bmatrix} \cos\theta(x_1+v_1) - \sin\theta(x_2+v_2) \\ \sin\theta(x_1+v_1) + \cos\theta(x_2+v_2) \end{bmatrix} = \circledast$. But then, on the other

$t_{v'}\rho_\theta = t_{\rho_\theta(v)}\rho_\theta = t_{\rho_\theta(v)}\rho_\theta\vec{x} = t_{\rho_\theta(v)} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t_{\rho_\theta(v)} \begin{bmatrix} \cos\theta x_1 - \sin\theta x_2 \\ \sin\theta x_1 + \cos\theta x_2 \end{bmatrix}$

But $\rho_\theta(v) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \cos\theta v_1 - \sin\theta v_2 \\ \sin\theta v_1 + \cos\theta v_2 \end{bmatrix}$, so

$t_{\rho_\theta(v)} \begin{bmatrix} \cos\theta x_1 - \sin\theta x_2 \\ \sin\theta x_1 + \cos\theta x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta x_1 - \sin\theta x_2 \\ \sin\theta x_1 + \cos\theta x_2 \end{bmatrix} + \begin{bmatrix} \cos\theta v_1 - \sin\theta v_2 \\ \sin\theta v_1 + \cos\theta v_2 \end{bmatrix} = \begin{bmatrix} \cos\theta(x_1+v_1) - \sin\theta(x_2+v \\ \sin\theta(x_1+v_1) + \cos\theta(x_2+v \end{bmatrix} = \star$

So $\circledast = \star$, we have verified the rule $\rho_\theta t_v = t_{v'}\rho_\theta$, where $v' = \rho_\theta(v)$.

(ii) w.t.s. $r t_v \stackrel{?}{=} t_{v'} r$, where $v' = r(v)$. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. On the one ha

$r t_v(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1+v_1 \\ x_2+v_2 \end{bmatrix} = \begin{bmatrix} x_1+v_1 \\ -x_2-v_2 \end{bmatrix} = \circledast$. On the oth

$t_{v'} r(\vec{x}) = t_{v'} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t_{v'} \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$,

But $t_{v'} = t_{r(v)}$, $r(v) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} \Rightarrow t_{v'} = t_{\left[\begin{smallmatrix} v_1 \\ -v_2 \end{smallmatrix}\right]}$, So

$t_{v'} \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = t_{\left[\begin{smallmatrix} v_1 \\ -v_2 \end{smallmatrix}\right]} \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} = \begin{bmatrix} x_1+v_1 \\ -x_2-v_2 \end{bmatrix} = \star$

So $\circledast = \star$, we have verified the rule $r t_v = t_{v'} r$, where $v' = r(v)$.

(iii) w.t.s. $r\rho_\theta \stackrel{?}{=} \rho_{-\theta} r$. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. then, on the one hand

$r\rho_\theta(\vec{x}) = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = r \begin{bmatrix} \cos\theta x_1 - \sin\theta x_2 \\ \sin\theta x_1 + \cos\theta x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta x_1 - \sin\theta x_2 \\ \sin\theta x_1 + \cos\theta x_2 \end{bmatrix}$

$= \begin{bmatrix} \cos\theta x_1 - \sin\theta x_2 \\ -\sin\theta x_1 - \cos\theta x_2 \end{bmatrix} = \#$

on the other,

$\rho_{-\theta} r(\vec{x}) = \rho_{-\theta} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \rho_{-\theta} \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$

$= \begin{bmatrix} \cos(-\theta)x_1 + \sin(-\theta)x_2 \\ \sin(-\theta)x_1 - \cos(-\theta)x_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta)x_1 - \sin(\theta)x_2 \\ -\sin(\theta)x_1 - \cos(\theta)x_2 \end{bmatrix} = @$

by properties of sin
$\sin(-\theta) = -\sin(\theta)$.
$\cos(-\theta) = \cos(\theta)$.

So $\# = @$, we have verified $r\rho_\theta = \rho_{-\theta} r$

(iv) W.t.S. $t_v t_w \stackrel{?}{=} t_{v+w}$. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. then,

$$(t_v t_w)(\vec{x}) = t_v\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = t_v \begin{bmatrix} w_1+x_1 \\ w_2+x_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1+x_1 \\ w_2+x_2 \end{bmatrix} = \begin{bmatrix} v_1+w_1+x_1 \\ v_2+w_2+x_2 \end{bmatrix} = \begin{bmatrix} v_1+w_1 \\ v_2+w_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= t_{v+w}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t_{v+w}(\vec{x}) \Rightarrow t_v t_w = t_{v+w}$$

(v) W.t.S. $\rho_\theta \rho_\eta \stackrel{?}{=} \rho_{\theta+\eta}$. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. then,

$$(\rho_\theta \rho_\eta)(\vec{x}) = \rho_\theta\left(\rho_\eta \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \rho_\theta\left(\begin{bmatrix} \cos\eta & -\sin\eta \\ \sin\eta & \cos\eta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \rho_\theta \begin{bmatrix} \cos\eta\, x_1 - \sin\eta\, x_2 \\ \sin\eta\, x_1 + \cos\eta\, x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\eta\, x_1 - \sin\eta\, x_2 \\ \sin\eta\, x_1 + \cos\eta\, x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta(\cos\eta\, x_1 - \sin\eta\, x_2) - \sin\theta(\sin\eta\, x_1 + \cos\eta\, x_2) \\ \sin\theta(\cos\eta\, x_1 - \sin\eta\, x_2) + \cos\theta(\sin\eta\, x_1 + \cos\eta\, x_2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\eta\, x_1 - \cos\theta\sin\eta\, x_2 - \sin\theta\sin\eta\, x_1 - \sin\theta\cos\eta\, x_2 \\ \sin\theta\cos\eta\, x_1 - \sin\theta\sin\eta\, x_2 + \cos\theta\sin\eta\, x_1 + \cos\theta\cos\eta\, x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1(\cos\theta\cos\eta - \sin\theta\sin\eta) - x_2(\cos\theta\sin\eta + \sin\theta\cos\eta) \\ x_1(\sin\theta\cos\eta + \cos\theta\sin\eta) + x_2(\cos\theta\cos\eta - \sin\theta\sin\eta) \end{bmatrix}$$

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$$= \begin{bmatrix} x_1\cos(\theta+\eta) - x_2\sin(\theta+\eta) \\ x_1\sin(\theta+\eta) + x_2\cos(\theta+\eta) \end{bmatrix}$$ By Properties of sine and cosine functions

$$= \begin{bmatrix} \cos(\theta+\eta) & -\sin(\theta+\eta) \\ \sin(\theta+\eta) & \cos(\theta+\eta) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (\rho_{\theta+\eta})(\vec{x}) \Rightarrow \rho_\theta\rho_\eta = \rho_{\theta+\eta}$$

(vi) W.t.S $rr = 1$. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. then,

$$rr(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\vec{x} = \vec{x} \Rightarrow rr(\vec{x}) = 1, \text{ i.e., } rr = id.$$

Q.3: (3.2) Let m be an orientation-reversing isometry. Prove algebraically that m^2 is a translation.

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Pf: Let $m: x \mapsto Ax+b$, where A is a reflection and b is a translation, be an arbitrary orientation-reversing isometry. Hence $mx = Ax+b$. Let us compute m^2: $m^2(x) = m(mx) = m(Ax+b)$
$= A(Ax+b)+b = A^2x + Ab + b = x + (Ab+b)$. because A is a reflection Hence $A^2 = I$. therefore,
m^2 = translation by a vector $Ab+b \Rightarrow m^2 = t_{Ab+b}$.

6.3 (3.3) Prove that a linear operator on $\mathbb{R}^2$ is a reflection if and only if its eigenvalues are 1 and -1, and the eigenvectors with these eigenva[lues] are orthogonal.

Pf: ($\Rightarrow$) Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be a linear operator on $\mathbb{R}^2$. Moreover, assume T is a reflection along a line ℓ. We want to show that: (i) eigenvalue[s] of T are 1 and -1 (ii) the eigenvectors with these eigenvalues are orthogonal.

(i) First, note that T fixes $\vec{x}$, $\forall \vec{x}\in\ell$: $T(\vec{x})=\vec{x} \Rightarrow 1$ is an eigenva[lue] of T. Moreover, let $\ell^\perp$ be the only orthogonal line to ℓ and $\vec{y}\in\ell^\perp$, then $T(\vec{y})=-\vec{y} \Rightarrow -1$ is the other eigenvalue of T. Since T acts on $\mathbb{R}^2$ it can only possibly have at most two distinct eigenvalues, so 1, -1, cover all possibilities.

(ii) Let $\vec{x}$ be an eigen vector with eigen value 1. then $T(a\vec{x})=a\vec{x}$, $\forall a\in$ [$\mathbb{R}$]. Let $\vec{y}$ be an eigenvector with eigenvalue -1. then $T(b\vec{y})=-b\vec{y}$. But by properties of T we must have $a\vec{x}\cdot b\vec{y}=0$, since $\vec{x}$ is a vector in ℓ and $\vec{y}$ is a vector in $\ell^\perp$, so they are orthogonal.

($\Leftarrow$) Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be a linear operator on $\mathbb{R}^2$. Moreover, assume that its only eigenvalues are 1 and -1 and the eigenvectors with these eigenval[ues] are orthogonal. We want to show that T is a reflection along a line.

Let $v\in\mathbb{R}^2$ be an eigen vector with eigen value 1, i.e. $T(v)=v$. Since T is linear, $aT(v)=av \Leftrightarrow T(av)=av$, $\forall a\in\mathbb{R}$. This means that the eigenvector with eigenvalue 1 spans a line that is unaffected by T. Likewise, let $w\in\mathbb{R}^2$ be an eigenvector with eigenvalue -1, i.e., $T(w)=-w$. Again, T linear implies $bT(w)=-bw \Leftrightarrow T(bw)=-bw$. Hence, this eigenvector span a line, which by hypothesis is orthogonal to v b/c $v\cdot w=0$, moreover, w is reflected along the line spanned by v. Therefore, T is a reflection along the line passing through $(0,0)$ in the direction of v.

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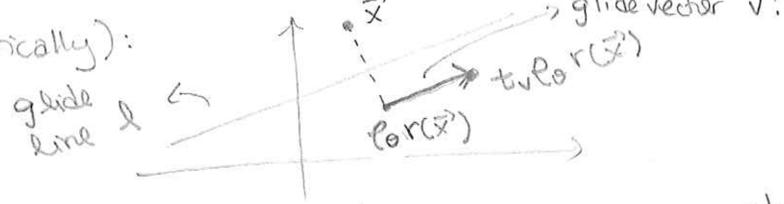
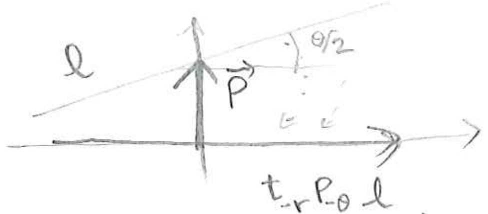
(1) Determine the glide line and glide vector for the glide reflection $t_v\rho_\theta r$. What conditions on v and θ are needed to make the glide reflection just a reflection?

Solution: Let $\theta\in\mathbb{R}$, $\vec{v}\in\mathbb{R}^2$. Consider the glide reflection $t_v\rho_\theta r$:

(Schematically):



We know that $\rho_\theta r$ is the reflection about the line at angle $\theta/2$ to x-axis. Consider the change of coordinate where we translate the glide line to the x-axis:



$\vec{p}$ is the position vector: $\vec{p}=\begin{bmatrix}0\\b\end{bmatrix}$, for some $b\in\mathbb{R}$.

then $m=\rho_\theta t_{\vec{p}}\, r\, t_{-\vec{p}}\rho_{-\theta}$. Now m is the glide reflection by first taking the glide line to the x-axis, reflecting, and then moving back to the original line. Now, with this characterization is easy to see that the glide line is given by $\ell(s)=\vec{p}+s\rho_\theta r(e_1)=\vec{p}+s\begin{bmatrix}\cos(\frac{\theta}{2})\\ \sin(\frac{\theta}{2})\end{bmatrix}$, where $s\in\mathbb{R}$. The glide vector is parallel to $\vec{v}$ in the description $t_v\rho_\theta r$, or equivalently, is a vector parallel to the glide line, i.e., a director vector of ℓ. Hence $\vec{v}=\begin{bmatrix}\cos(\frac{\theta}{2})\\ \sin(\frac{\theta}{2})\end{bmatrix}$.

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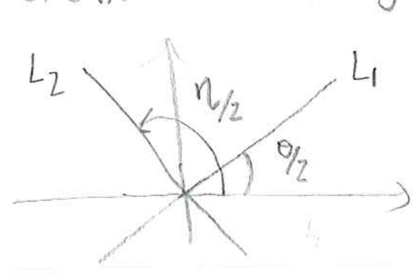
The conditions on $\vec{v}$ and θ needed to make the glide reflection just a reflection are: $r=\rho_\theta t_{\vec{p}}\, r\, t_{-\vec{p}}\rho_{-\theta}$; i.e. $\vec{v}=0$ and θ can take any value: $r=\rho_\theta t_{\vec{0}}\, r\, t_{-\vec{0}}\rho_{-\theta}=\rho_\theta r\rho_{-\theta}=\rho_\theta\rho_\theta r=\rho_{2\theta}r$.

(2) Prove that if L_1 and L_2 are lines through the origin in $\mathbb{R}^2$ then the composition of the reflections across the two lines is a rotation and determine the angle of rotation

Pf:



Consider the reflections across L_1 and L_2 given by $\rho_\theta r$ and $\rho_\eta r$. Let us compose this isometries and analyze the result:

$(\rho_\theta r)(\rho_\eta r) = \rho_\theta((r\rho_\eta)r) = \rho_\theta((\rho_{-\eta}r)r)$ By rules proved in exercise 6.3 (3.1)

$= (\rho_\theta \rho_{-\eta})(rr)$

$= \rho_\theta \rho_{-\eta} 1$

$= \boxed{\rho_{\theta-\eta}}$

therefore, the composition of the reflections across L_1, L_2 is a rotation with angle $\theta-\eta$; where $\frac{\theta}{2}$ is the angle of the first line w.r.t the x-axis and $\frac{\eta}{2}$ is the angle of the second line w.r.t the x-axis. Since L_1, L_2 are arbitrary lines through the origin, we get the result.

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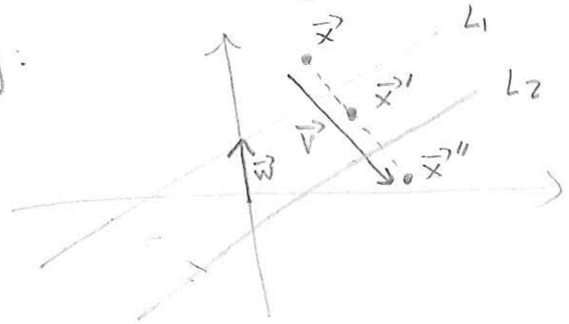
(3) SHOW the composition of reflections about parallel lines is a translation by a vector orthogonal to the lines

Pf: Geometrically:



reflection about line L_1 sends $\vec{x}$ to $\vec{x}'$. Next, reflection about L_2 sends $\vec{x}'$ to $\vec{x}''$. One can see that $\vec{x}''$ is a translation of $\vec{x}$ by vector $\vec{v}$ which is orthogonal to both L_1 and L_2.

Algebraically, let us represent a reflection about an arbitrary line as $m_1 = t_{w_1}\rho_\theta r \rho_{-\theta} t_{-w_1}$, where $\vec{w}$ is a vector parallel to the y-axis pointing to a point on the line. A parallel line has the same angle θ but possibly differs on the vector $\vec{w}$. So let $m_2 = t_{w_2}\rho_\theta r \rho_{-\theta} t_{-w_2}$. Finally, let us compose these two isometries:

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$m_1 \circ m_2 = (t_{w_1}\rho_\theta r \rho_{-\theta} t_{-w_1})(t_{w_2}\rho_\theta r \rho_{-\theta} t_{-w_2})$ By rules previously proved

$= t_{w_1}\rho_\theta r \rho_{-\theta} t_{w_2-w_1} \rho_\theta r \rho_{-\theta} t_{-w_2}$

$= t_{w_1} r \rho_{\theta-\theta} t_{w_2-w_1} r \rho_{\theta-\theta} t_{-w_2}$

$= t_{w_1} r t_{w_2-w_1} r t_{-w_2}$

$= t_{w_1} t_z r r t_{-w_2}$, $z = r(w_2-w_1)$

$= t_{w_1} t_z t_{-w_2}$

$= t_z t_{w_1-w_2}$

$= t_{r(w_2-w_1)+(w_1-w_2)} = t_{w_1-w_2+w_1-w_2} = t_{2(w_1-w_2)}$, showing the result.