(1)

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(1) (a). Let S,T be sets and let $f: S \to T$ be a function. Define a relation $\sim$ on S by $s_1 \sim s_2$ if $f(s_1)=f(s_2)$. Prove that $\sim$ is an equivalence relation.

Pf: Need to prove: reflexivity, symmetry, and transitivity.

(i) Reflexivity: Let $s \in S$. Certainly: $s=s$ and since f is a function, $f(s)=f(s)$. therefore $s \sim s$ for any $s \in S$. So $\sim$ is reflexive.

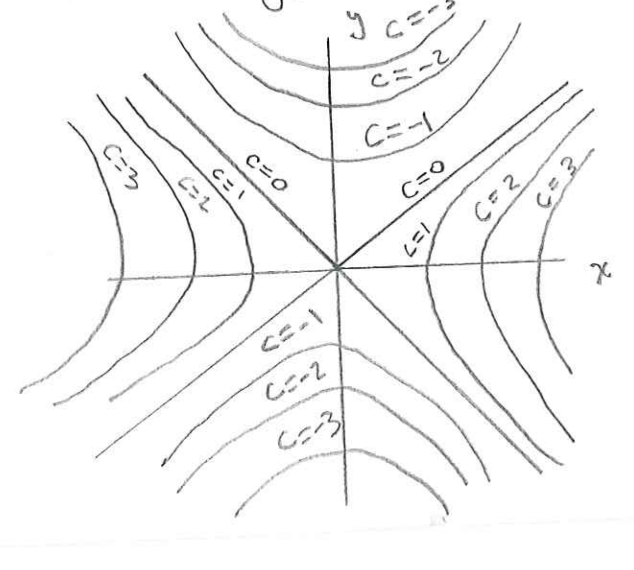
(ii) Symmetry: Let $s_1, s_2 \in S$. Suppose that $s_1 \sim s_2$. then $f(s_1)=f(s_2)$ which is the same as $f(s_2)=f(s_1)$ and so $s_2 \sim s_1$. thus, $\sim$ is symmetri

(iii) Transitivity: Let $s_1, s_2, s_3 \in S$. Suppose that $s_1 \sim s_2$ and $s_2 \sim s_3$. then, $f(s_1)=f(s_2)$ and $f(s_2)=f(s_3)$. Replacing the second equction on the first we get $f(s_1)=f(s_3)$. So $s_1 \sim s_3$. therefore, $\sim$ is transitive. +5

(i), (ii) and (iii) imply that $\sim$ is an equivalence relation.

(1)(b) Let $S=\mathbb{R}^2$ and $T=\mathbb{R}$. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y)=x^2-$
Draw the equivalence classes for the equivalence relation determined (as i part (a)) by f. +5

Solution: Let (x_1,y_1) be a point in $\mathbb{R}^2$. By definition of $\sim$, its equivale class is: $[(x_1,y_1)] = \{(x,y) \in \mathbb{R}^2 \mid (x_1,y_1) \sim (x,y) \Leftrightarrow f(x_1,y_1)=f(x,y) \Leftrightarrow x_1^2-y_1^2=x^2-$ thinking of (x_1,y_1) as arbitrary but fixed, its equivalence class is just th level set of the hyperbolic paraboloid $f(x,y)=x^2-y^2$. In picture:



when $c=0$, we have two lines $x=y$ and $-x=y$.

when $c>0$, we have a hyperb opening on the x-axis

when $c<0$, we have a hype opening on the y-axis

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(2) Determine all subgroups of D_4.

Solution: By Lagrange's theorem we know that the order of a subgr[oup] in D_4 must divide $|D_4| = 8$. Hence, the only possible groups are of order: 1, 2, 4 and 8. Moreover, the only groups of order 1 and 8 are the trivial subgroup and D_4 itself respectively. Therefore, it makes sense to look only for groups of order 2 and 4.

By inspection all the groups of order 2 are:

$\{I, R_2\}, \{I, D_1\}, \{I, D_2\}, \{I, H\}, \{I, V\}$, since all other elements operated with themselves produce something other than I (Look at th[e] diagonal of the group table).

It remains to determine all subgroups of order 4. We can look f[or] generators of such groups as follow:

$\langle R_1 \rangle = \{I, R_1, R_1^2, R_1^3\} = \{I, R_1, R_2, R_3\}$, all rotations.

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We can also obtain the subgroups:

$\{I, R_2, D_1, D_2\}$ and $\{I, R_2, H, V\}$.

Is easy to see that these 10 subgroups are all the subgroups of D[4]
$\{I\}, \{I, R_2\}, \{I, D_1\}, \{I, D_2\}, \{I, H\}, \{I, V\}, \{I, R_2, D_1, D_2\}, \{I, R_2, H, V\}, \{I, R_1, R_2, R_3\}$,

If you try to compute any other subgroup, it will be one of these, e.g.
$\{R_3, H\} \Rightarrow R_3H = D_2$; $R_3R_3 = R_2$; $R_2D_2 = D_1$; so far we have
$\{R_3, H, D_2, R_2, D_1\}$, but this is already 5 elements, $5 \nmid 8$, so if u keep computing these elements we will get D_4. Likewise, start wi[th]
$\{R_3, D_1\} \Rightarrow R_3D_1 = H$; $R_3R_3 = R_2$; $R_2D_1 = D_2$; so far we have
$\{R_3, D_1, H, R_2, D_2\}$, using the same reasoning as before, this set will eventually be D_4. In this manner we can check that ind[eed]
D_4 has only the 10 elements shown before.

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(3) Let $m, n \in \mathbb{Z}$. We have proved there is a unique integer l such that $m\mathbb{Z} \cap n\mathbb{Z} = l\mathbb{Z}$. Prove that l is the least common multiple of m, n

Pf: Without loss of generality, we may assume that $m, n, l > 0$. Note that if either $m=0$ or $n=0$ the result follows trivially since $\{0\} = 0\mathbb{Z} \cap 0\mathbb{Z} = 0\mathbb{Z}$, so 0 is the l.c.m of $0, 0$. Moreover, if either $m<0$ or $n<0$, we can work with $-m\mathbb{Z}$ or $-n\mathbb{Z}$ respectively since $-m\mathbb{Z} = m\mathbb{Z}$ and $-n\mathbb{Z} = n\mathbb{Z}$.

Now, by definition $l \in l\mathbb{Z}$ which means that $l \in m\mathbb{Z} \cap n\mathbb{Z}$ and so $l \in m\mathbb{Z}$, $l \in n\mathbb{Z}$. Again, by definition $m | l$ and $n | l$ and so l is a multiple of both m and n, so it is a common multiple. -10 To show that l is the least common multiple, let $x \in \mathbb{Z}, x > 0$, such that $m | x$ and $n | x$. By definition, $x \in m\mathbb{Z} \cap n\mathbb{Z}$ and thus $x \in l\mathbb{Z}$ and so $l | x$, i.e., $x = l \cdot q$, so $q > 0$, $l \leq x$. So l the least common multiple.//

(4) Let H and K be subgroups of a group G.
Prove that $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.

Pf: ($\Rightarrow$) Let $H \cup K \leq G$. We want to prove that $H \subseteq K$ or $K \subseteq H$. To the contrary, suppose that $H \not\subseteq K$ and $K \not\subseteq H$, then, there exists elements: $h \in H \setminus K$ and $k \in K \setminus H$. Look at hk. Since $H \cup K$ is a subgroup of G, it must be that $hk \in H \cup K$ ($h \in H \Rightarrow h \in H \cup K$; likewise $k \in K \Rightarrow k \in H \cup K$). By definition, $hk \in H$ or $hk \in K$.

If $hk \in H$ then $h^{-1}(hk) = (h^{-1}h)k = ek = k \in H$; Since $h^{-1}, hk \in H$. Contradiction

If $hk \in K$ then $(hk)k^{-1} = h(kk^{-1}) = he = h \in K$; Since $k^{-1}, hk \in K$. Contradiction

In any case we reach a contradiction.

Therefore, $H \subseteq K$ or $K \subseteq H$.

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(4) ($\Leftarrow$) Suppose that $H \subseteq K$ or $K \subseteq H$. We want to prove that $H \cup K \overset{?}{\leq} G$

(i) Let $h, k \in H \cup K$. then we have the following cases:

- $h \in H$ and $k \in H$. then $hk \in H$ since H is a subgroup, so it is closed
- $h \in K$ and $k \in K$. then $hk \in K$ since K is a subgroup, so it is closed
- $h \in H$ and $k \in K$. then If $H \subseteq K$ then $h \in K$ and so $hk \in K$
  Otherwise, If $K \subseteq H$ then $k \in H$ and so $hk \in H$

this final case is symmetrical with $h \in K$ or $k \in H$, so it holds in that case too.
Note that the statement $h \in H$ implies that $h \in H \cup K$, and $k \in K \Rightarrow k \in H \cup K$
therefore, $H \cup K$ is closed under the operation of G.

(ii) Let $h \in H \cup K$. then $h \in H$ or $h \in K$. If $h \in H$ then $h^{-1} \in H$, since H is a group. Otherwise, if $h \in K$ then $h^{-1} \in K$, since K is a group.
In any case $h^{-1} \in H \cup K$.

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Since (i) and (ii) hold, we conclude that $H \cup K$ is a subgroup of G.

(5) the relation $\sim$ on S is an equivalence relation:

(i) Reflexivity: Let $s \in S$. take $k = 0$. then $f^k(s) = f^0(s) = Id(s) = s$. then $s \sim s$

(ii) Symmetry: Let $s_1, s_2 \in S$ be such that $s_1 \sim s_2$. then, there exists $k \in \mathbb{Z}$ such that $f^k(s_1) = s_2$. Apply f^{-k} to both sides of this equation:

$$f^{-k}(f^k(s_1)) = f^{-k}(s_2) \Rightarrow f^{-k+k}(s_1) = f^{-k}(s_2) \Rightarrow f^0(s_1) = s_1 = f^{-k}(s_2).$$

Hence, there exists an integer $-k$ such that $s_2 \sim s_1$

(iii) Transitivity: Let $s_1, s_2, s_3 \in S$, be such that $s_1 \sim s_2$ and $s_2 \sim s_3$. then, there exists integers k, l such that: $f^k(s_1) = s_2$ and $f^l(s_2) = s_3$
Apply f^{-l} to both sides of the last equation: $f^{-l}(f^l(s_2)) = f^{-l}(s_3) \Rightarrow s_2 = f^{-l}(s_3)$. Replace s_2 in the first equation: $f^k(s_1) = s_2 = f^{-l}(s_3) \Rightarrow f^k(s_1) = f^{-l}(s_3)$. Finally, apply f^l to both sides of this equation to get
$f^l(f^k(s_1)) = f^l(f^{-l}(s_3)) \Rightarrow f^{l+k}(s_1) = s_3$. So there exist an integer $l + k = t$ such that $f^t(s_1) = s_3 \Leftrightarrow s_1 \sim s_3$.

(6)(a) Let $H \leq G$. Define a relation $\sim$ on G by $g_1 \sim g_2$ if $g_1 g_2^{-1} \in H$. $\sim$ is an equivalence relation.

Pf: (i) Reflexivity: Let $g \in G$. By definition of subgroup we know that $gg^{-1} = e \in H$, for any $g \in G$. Hence, $\sim$ is reflexive.

(ii) Symmetry: Let $g_1, g_2 \in G$. Suppose that $g_1 \sim g_2$. then $g_1 g_2^{-1} \in H$. By definition of subgroup, this element has an inverse in H, i.e.,

$(g_1 g_2^{-1})^{-1} \in H \iff (g_2^{-1})^{-1} g_1^{-1} \in H \iff g_2 g_1^{-1} \in H \iff g_2 \sim g_1$.

(iii) Transitivity: Let $g_1, g_2, g_3 \in G$. Suppose that $g_1 \sim g_2$ and $g_2 \sim g_3$. then $g_1 g_2^{-1} \in H$ and $g_2 g_3^{-1} \in H$. By definition of subgroup: $(g_1 g_2^{-1})(g_2 g_3^{-1}) \in H$

$\iff g_1[(g_2^{-1} g_2) g_3^{-1}] = g_1(e g_3^{-1}) = g_1 g_3^{-1} \in H \iff g_1 \sim g_3$ □

the equivalence classes are precisely the right cosests Hg for $g \in G$.

Pf: By definition; given $g \in G$ its equivalence class is:

$[g] = \{x \in G \mid x \sim g \iff xg^{-1} \in H\}$. We want to show that $[g] = Hg$.

($\subseteq$) Let $a \in [g]$. then $a \sim g \iff ag^{-1} \in H$. let $h = ag^{-1} \in H$. then, apply g to both sides of this equation $hg = a(g^{-1}g) \Rightarrow hg = a$. therefore $a \in Hg$, since there exists $h \in H$, $h = ag^{-1}$, such that $hg = a$.

($\supseteq$) Let $a \in Hg$ then, there exists $h \in H$ such that $a = hg$. Apply g^{-1} to both sides of this equation to get $ag^{-1} = h \in H$. therefore $a \sim g$ which means that $a \in [g]$.

(b). Let $G = D_4$. Let $H = \{I, H\} \leq G = D_4$. Pick the elements $g_1 = R_1$ and $g_2 = D_1$. then

$g_1 H = R_1 H = \{R_1 I, R_1 H\} = \{R_1, D_1\} = \{D_1 I, D_1 H\} = D_1 H = g_2 H$

But,

$Hg_1 = HR_1 = \{IR_1, HR_1\} = \{R_1, D_2\}$

$Hg_2 = HD_1 = \{ID_1, HD_1\} = \{D_1, R_3\}$

$\Rightarrow Hg_1 \neq Hg_2$.

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(6)(c). Let H be a subgroup of a group G and let $g_1, g_2 \in G$. Prove that $g_1H = g_2H$ if and only if $Hg_1^{-1} = Hg_2^{-1}$.

Pf: ($\Rightarrow$). Suppose that $g_1H = g_2H$. We want to show: $Hg_1^{-1} \stackrel{?}{=} Hg_2^{-1}$.

First note that: $x \in g_1H \Leftrightarrow x \in g_2H$, hence, let $x \in g_1H$. then $x = g_1h_1$ and $x = g_2h_2$, for some $h_1, h_2 \in H$.

But then $g_1h_1 = g_2h_2 \begin{cases} \Rightarrow g_1 = g_2h_2h_1^{-1} \Rightarrow g_1^{-1} = h_1h_2^{-1}g_2^{-1} \quad (*) \\ \Rightarrow g_2 = g_1h_1h_2^{-1} \Rightarrow g_2^{-1} = h_2h_1^{-1}g_1^{-1} \quad (**) \end{cases}$

($\subseteq$) Let $x \in Hg_1^{-1}$. then $x = hg_1^{-1}$, for some $h \in H$. Replacing (*)

$x = hg_1^{-1} = h(h_1h_2^{-1}g_2^{-1}) = (hh_1h_2^{-1})g_2^{-1}$; since $h, h_1, h_2^{-1} \in H$,

Let $h_3 = hh_1h_2^{-1} \in H$. We have found $h_3 \in H$ s.t $x = h_3g_2^{-1}$.

therefore $x \in Hg_2^{-1}$.

($\supseteq$) Let $x \in Hg_2^{-1}$. then $x = hg_2^{-1}$, for some $h \in H$. Replacing (**)

$x = hg_2^{-1} = h(h_2h_1^{-1}g_1^{-1}) = (hh_2h_1^{-1})g_1^{-1}$; since $h, h_2, h_1^{-1} \in H$,

Let $h_4 = hh_2h_1^{-1} \in H$. We have found $h_4 \in H$ s.t. $x = h_4g_1^{-1}$

therefore $x \in Hg_1^{-1}$.

Hence, $Hg_1^{-1} = Hg_2^{-1}$.

($\Leftarrow$) Suppose that $Hg_1^{-1} = Hg_2^{-1}$. We want to show: $g_1H = g_2H$

First note that: $x \in Hg_1^{-1} \Leftrightarrow x \in Hg_2^{-1}$, hence, let $x \in Hg_1^{-1}$. then: $x = h_1g_1^{-1}$ and $x = h_2g_2^{-1}$, for some $h_1, h_2 \in H$.

But then $h_1g_1^{-1} = h_2g_2^{-1} \begin{cases} \Rightarrow g_1^{-1} = h_1^{-1}h_2g_2^{-1} \Rightarrow g_1 = (g_1^{-1})^{-1} = g_2h_2^{-1}h_1 \quad (*) \\ \Rightarrow g_2^{-1} = h_2^{-1}h_1g_1^{-1} \Rightarrow g_2 = (g_2^{-1})^{-1} = g_1h_1^{-1}h_2 \quad (**) \end{cases}$

Using a similar argument as that used for ($\Rightarrow$). We have:

($\subseteq$) Let $x \in g_1H$. then $x = g_1h$, for some $h \in H$. Replacing (*)
$x = g_1h = g_2(h_2^{-1}h_1h)$; since $h, h_1, h_2^{-1} \in H$, let $h_5 = h_2^{-1}h_1h_2 \in H$.
We have found $h_5 \in H$ s.t $x = g_2h_5$. therefore, $x \in g_2H$.

($\supseteq$) Let $x \in g_2H$. then $x = g_2h$, for some $h \in H$. Replacing (**)
$x = g_2h = g_1(h_1^{-1}h_2h)$; since $h, h_1^{-1}, h_2 \in H$, let $h_6 = h_1^{-1}h_2h \in H$.
We have found $h_6 \in H$ s.t $x = g_1h_6$. therefore, $x \in g_1H$.
Hence, $g_1H = g_2H$.

(7) Let G be a group in which for all $g \in G$, $g^2 = e$.
Prove G is abelian.

Pf: Let $g_1, g_2 \in G$. then

$$
\begin{aligned}
g_1g_2 &= e\,g_1\,g_2\,e && \text{By properties of identity elem} \\
&= (g_2g_2)(g_1g_2)(g_1g_1) && \text{Letting } g_2g_2 = e \text{ and } g_1g_1 = e \\
&= g_2[(g_2g_1)(g_2g_1)]g_1 && \text{Associativity of the group} \\
&= g_2(g_2g_1)^2g_1 && \text{Power notation} \\
&= g_2\,e\,g_1 && \text{By hypotesis} \\
&= g_2g_1 && \text{By properties of identity elem}
\end{aligned}
$$

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therefore, $g_1g_2 = g_2g_1$ for any $g_1, g_2 \in G$.
G is abelian.