M403 - Fall 2013 - Enrique Areyan - HW4

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(1) Let G be a group.

For parts (a) and (b), let us first show a preliminary result.

Proposition: Let G be a group with a finite number of subgroups. then G is fini

Pf: By contraposition, suppose that G is an infinite group.

If G is cyclic then $G \cong (\mathbb{Z},+)$ and since $(\mathbb{Z},+)$ have infinitely many subgro

so does G (isomorphisms preserve subgroups).

Otherwise, if G is not cyclic; take $g \in G$, $g \neq e$ and look at $\langle g \rangle \neq G$.

if $\langle g \rangle$ is infinite then $\langle g \rangle \cong (\mathbb{Z},+)$, and so by the same argument as befor

and the fact that $\langle g \rangle \subset G$; we conclude that G has infinitely many subgro

otherwise, if $\langle g \rangle$ is finite, take $g_1 \in G \setminus \langle g \rangle$. Repeat the argument looking now

$\langle g_1 \rangle$. Hence, we either get that one of $g, g_1, g_2, \ldots, g_n, \ldots$ generates and

infinite subgroups or each of these generate its own subgroup $\langle g_i \rangle \neq \langle g_j \rangle$,

$i \neq j$, $i,j = 1,2,\ldots$ In either case G has infinitely many subgroups.

(a) Prove that if G has exactly three subgroups, then G is finite cyclic and $|G| = p^2$ for some prime p.

Pf: Let G be a group with exactly three subgroups. By previous proposition, there
G is finite. By definition, $\langle e \rangle$ and G itself are subgroups of G. there
G has only one other proper subgroup, call it $H \subset G$. Now, take $g \in G \setminus H$ an
look at $\langle g \rangle$. this subgroup has to be one of $\langle e \rangle, H, G$. But it cannot
be $\langle e \rangle$ since $e \notin G \setminus H$. It cannot be H since $g \notin H$. therefore $\langle g \rangle = G$, w
shows that G is cyclic. Moreover, let $|G| = n$. In class we proved that
finite cyclic group of order n is such that for every $m | n$, there is exac
one subgroup of order m. Since G is cyclic with three subgroups the
n is divisible only by $1, n, q$; where $1 < q < n$. But the only numbers wit
exactly three divisors are p^2 for a prime p, for otherwise suppose
$n = a.b$ for positive integers a and b. then $1|n$, $n|n$ but $a|n$ and $b|$
so our cyclic subgroup would have four instead of three subgroups.
therefore, $|G| = p^2$ for p a prime.

(b) Prove that if G has exactly four subgroups, then G is finite cy
and $|G|$ is either p^3 for some prime p or pq for distinct primes p a

Pf: Following a similar argument as before: By previous proposition G is f

By definition $\langle e \rangle$, G are subgroups of G. Hence, there exists subgroups H_1, H_2, such that $H_1 \neq H_2$, and $H_1 \subset G$, $H_2 \subset G$. Now, take $g \in G \setminus H_1 \cup H_2$ and look at $\langle g \rangle$. this subgroup has to be one of $\langle e \rangle, H_1, H_2$ or G. But it cannot be $\langle e \rangle$ since $e \notin G \setminus H_1 \cup H_2$. It cannot be either one of H_1, H_2 since $g \notin H_1$ and $g \notin H_2$. therefore $\langle g \rangle = G$, which shows that G is cyclic. So, we have G a finite cyclic group. By proposition proved in class, for every divisor $m | n = |G|$ there is exactly one subgroup of order m. But we have only four subgroups so n has to be divisible only by four numbers: $1, n, r, s$. So either $n = p^3$ for p a prime in which case $1 | p^3$, $p^3 | p^3$, $p^2 | p^3$, $p | p^3$, so that $|\langle e \rangle| = 1$, $|G| = p^3$, $|H_1| = p^2$ and $|H_2| = p$ OR $n = pq$ for distinct primes p and q in which case $1 | pq$, $pq | pq$, $p | pq$, $q | pq$, so that $|\langle e \rangle| = 1$, $|G| = pq$, $|H_1| = p$ and $|H_2| = q$.

No other combination will work for suppose $|G| = n = p.a$, for p a prime and a an integer. then we can write $n = p(qr)$; for primes q, r, and so we will get more than four subgroups since $1 | pqr$, $pqr | pqr$, $p | pqr$, $q | pqr$, $r | pqr$, a contradiction.

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Therefore, $|G| = p^3$ for p a prime or $|G| = pq$, for $p, \cancel{q}$ distinct primes.

(2) Let G be a (possibly infinite) group. Let H be a subgroup.

(a) Prove that $\tilde{H} = \bigcap_{g \in G} gHg^{-1}$ is a normal subgroup of G.

Pf: Let us prove that $\forall g \in G: g\tilde{H}g^{-1} = \tilde{H}$; and thus conclude that $\tilde{H}$ is normal. Let $g \in G$. Let $x \in g\tilde{H}g^{-1}$. $\Leftrightarrow x \in g\left[\bigcap_{g' \in G} g'Hg'^{-1}\right]g^{-1} \Leftrightarrow x \in \bigcap_{g' \in G} gg'Hg'^{-1}g^{-1} \Leftrightarrow x \in \bigcap_{g' \in G} gg'H(gg')^{-1}$

Now, we proved in class that $x \mapsto gxg^{-1}$, conjugation by g is an automorphism. In particular it is 1-1 and onto. therefore,

$\Leftrightarrow x \in \bigcap_{g \in G} gg'H(gg')^{-1} \Leftrightarrow x \in \bigcap_{g' \in G} g'Hg'^{-1} \Leftrightarrow x \in \tilde{H}$. therefore $\tilde{H}$ is normal.

Prove that $\tilde{H}$ is the largest normal subgroup of G contained in H, i.e., If K is any normal subgroup of G s.t. $K \subseteq H$ then $K \subseteq \tilde{H}$.

Pf: Let $K \trianglelefteq G$ and $K \subseteq H$. By definition of normality, $\forall g \in G: \forall k \in K: gkg^{-1} \in K$

Let $\kappa \in K$. then $g\kappa g^{-1} \in K$ for any $g \in G$. But any element of K is an element H and so $g\kappa g^{-1} \in H$, which means that there exist $h \in H$ such that $h = g\kappa g^{-1} \Rightarrow \kappa = g^{-1}hg$; let $g' = g^{-1}$; then $\kappa = g'hg'^{-1}$, for any g'. Hence $\kappa \in \bigcap_{g' \in G} g'Hg'^{-1} \Rightarrow \kappa \in \hat{H}$.

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(b) Now suppose G contains a subgroup of finite index. Prove that G contains a normal subgroup of finite index.

Pf: Define the homomorphism $\varphi: G \to G/H$, where H is a subgroup of G of finite index. We showed in class that G/H with $\circ: G/H \to G/H$ defined as $(g_1H)\circ(g_2H) = g_1g_2H$ is a group. Moreover $\varphi: G \to G/H$ given by $\varphi(g) = gH$ is a homomorphism. We also proved that the kernel of a homomorphism is a normal subgroup of the domain group. In this case $Ker(\varphi) \trianglelefteq G$. But by definition $Ker(\varphi) = H$. Since H has finite index, so will $Ker(\varphi)$. So we have found a normal subgroup of finite index, namely $Ker(\varphi)$.

(3) Let G be a group and let H be a subgroup. Prove that if H has index two in G, then H is normal.

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Pf: Let $H \leq G$ be such that $[G:H] = 2$. this means that there are only two distinct left cosests of H in G. Taking $e \in G$, we know that $eH = H$ is left coset. therefore, the only two cosests of H in G are $H, g'H$ wh $g' \notin H$. We want to show that H is normal. But before we proceed, Let us first show that given $g', g'' \notin H$ then $g'g'' \in H$. Suppose for a contradiction that $g'g'' \notin H$. Since there are only two left cosest we have $g'g'' \in g'H \Rightarrow g'g'' = g'h$, for some $h \in H$. But then, operating by g'^{-1} we get $g'' = h \in H$; a contradiction since $g'' \notin H$.

Now, to show normality, let $g \in G$ and $h \in H$. then

If $g \in H$ then $ghg^{-1} = \boxed{(gh)g^{-1} \in H}$, since $g \in H, h \in H \Rightarrow gh \in H$. $\Rightarrow g^{-1} \in H$.

otherwise $g \notin H$. then $g = g'h \in g_1H \Rightarrow gh^{-1} = g' \notin H$. h^{-1} is some element in H so we can just write $gh \notin H$. But then, $ghg^{-1} = \boxed{(gh)g^{-1} \in H}$. Since $gh \notin H$ and $g^{-1} \notin H$, and by argument before the product of two elements not in H is in H. therefore, H is normal.

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(4) Find all the subgroups of C_{12}, the cyclic group of order 12.

Solution: C_{12} is finite cyclic. therefore, every subgroup is cyclic and for every $m|12$, there is exactly one subgroup of order m. the possible divisors of 12 are $m=1,2,3,4,6,12$. So, we know that C_{12} has exactly 6 subgroups. Let $C_{12}=\langle g\rangle$. then, $O(g)=12$.

Generators for each subgroups are:

$\langle g^0\rangle=\langle e\rangle=\langle g^{12}\rangle$

$\langle g\rangle=C_{12}=\langle g^5\rangle=\langle g^7\rangle=\langle g^{11}\rangle$

$\langle g^2\rangle=\{e,g^2,g^4,g^6,g^8,g^{10}\}=\langle g^{10}\rangle$

$\langle g^3\rangle=\{e,g^3,g^6,g^9\}=\langle g^9\rangle$

$\langle g^4\rangle=\{e,g^4,g^8\}=\langle g^8\rangle$

$\langle g^6\rangle=\{e,g^6\}$

this is an example of a general fact proved in class, namely: all elements generate some cyclic subgroup.

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(5) ~~In~~ D_4, Let $N=\langle R_2\rangle$, the subgroup generated by R_2. We have seen that N is normal. the quotient group D_4/N is a group you know. What is it?

Solution: By definition $D_4/N=\{xN \mid x\in D_4\}=\{x\langle R_2\rangle \mid x\in D_4\}$. Let us compute

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$x=I \Rightarrow I\langle R_2\rangle=\langle R_2\rangle=\{e,R_2\}$

$x=R_1 \Rightarrow R_1\langle R_2\rangle=\{R_1I,R_1R_2\}=\{R_1,R_3\}$

$x=R_2 \Rightarrow R_2\langle R_2\rangle=\{R_2I,R_2R_2\}=\{R_2,I\}$

$x=R_3 \Rightarrow R_3\langle R_2\rangle=\{R_3I,R_3R_2\}=\{R_3,R_1\}$

$x=D_1 \Rightarrow D_1\langle R_2\rangle=\{D_1I,D_1R_2\}=\{D_1,D_2\}$

$x=D_2 \Rightarrow D_2\langle R_2\rangle=\{D_2I,D_2R_2\}=\{D_2,D_1\}$

$x=H \Rightarrow H\langle R_2\rangle=\{HI,HR_2\}=\{H,V\}$

$x=V \Rightarrow V\langle R_2\rangle=\{VI,VR_2\}=\{V,H\}$

Hence, we can write

$D_4/N=\{\{I,R_2\},\{R_1,R_3\},\{D_1,D_2\},\{H,V\}\}$

So D_4/N is of order 4. Hence, it is isomorphic to either U_8 or $(\mathbb{Z}_4,\oplus)$

By inspecting the Cayley table of D_4/N we can easily conclude that $D_4/N\cong U_8$

	$\{I,R_2\}$	$\{R_1,R_3\}$	$\{D_1,D_2\}$	$\{H,V\}$
$\{I,R_2\}$	$\{I,R_2\}$	$\{R_1,R_3\}$	$\{D_1,D_2\}$	$\{H,V\}$
$\{R_1,R_3\}$	$\{R_1,R_3\}$	$\{I,R_2\}$	$\{H,V\}$	$\{D_1,D_2\}$
$\{D_1,D_2\}$	$\{D_1,D_2\}$	$\{H,V\}$	$\{I,R_2\}$	$\{R_1,R_3\}$
$\{H,V\}$	$\{H,V\}$	$\{D_1,D_2\}$	$\{R_1,R_3\}$	$\{I,R_2\}$

the explicit isomorphism is given by

$f: D_4/N \to U_8$

$f(\{I,R_2\})=1$

$f(\{R_1,R_3\})=3$

$f(\{D_1,D_2\})=5$

$f(\{H,V\})=7$

Every element in $D4/N$ has order 2

(6) Let $f: G_1 \to G_2$ be a group homomorphism and let $N = Ker(f)$. In class we showed that there is an induced homomorphism from G_1/N to G
Generalize this by showing that if K is a normal subgroup of G_1 such th
$K \subseteq N$, then there is an induced homomorphism from G_1/K to G_2.

Pf: Let $f: G_1 \to G_2$ be a group homomorphism. Let $N = Ker(f)$.
Let $K \trianglelefteq G_1$ such that $K \subseteq N$. the following diagram summarizes the informa
we have and what we wish to prove:

$$\begin{tikzcd} G_1 \arrow[r, "f"] \arrow[d, "\pi"'] & G_2 \\ G_1/K \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

We want to prove the existence of $\psi: G_1/K \to G_2$; and show that ψ is a homomorphism.

the map $\pi: G_1 \to G_1/K$ defined by $\pi(g) = gK$ was shown to be well defined and an homorphism in the case where $K = N$. this is just the cannonical ma
In this case, let us show that $\psi(gK) = f(g)$ is well defined and a homomor

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(i) Well defined: Suppose $gK = g_1K$. then

$$gK = g_1K \Rightarrow g^{-1}g_1 \in K \subseteq N. \Rightarrow f(g^{-1}g_1) = e \Rightarrow f(g^{-1})f(g_1) = e$$

$\Rightarrow f(g)^{-1}f(g_1) = e \Rightarrow f(g) = f(g_1)$. So it is well defined.

(ii) ψ is a homomorphism:

$$\psi(g_1Kg_2K) = \psi(g_1g_2K) = f(g_1g_2) = f(g_1)f(g_2) = \psi(g_1K)\psi(g_2K).$$

So ψ is a homomorphism.