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(1) Consider the following two matrices in $GL_2(\mathbb{C})$:

$$x=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\quad y=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\qquad \text{Let } z = xy = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \\ i & \end{pmatrix}$$

(a) SHOW that the set $Q_8 = \{\pm 1, \pm x, \pm y, \pm xy\}$, is a subgroup of $GL_2(\mathbb{C})$ and write out its group table.

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Solution:

(i) the group is closed under inverses:

$1^{-1} = 1$; $-1^{-1} = -1$; (follow from basic properties of the identity matrix).

$x^{-1} = -x$, since $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & +i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -i & 0 \\ 0 & +i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow -x^{-1} = x$

$y^{-1} = -y$, since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow -y^{-1} = y$

$z^{-1} = -z$, since $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow -z^{-1} = z$

(ii) the following group table shows that Q_8 is closed under matrix multiplication and thus, it is a subgroup of $GL_2(\mathbb{C})$.

	1	-1	x	-x	y	-y	xy	-xy
1	1	-1	x	-x	y	-y	xy	-xy
-1	-1	1	-x	x	-y	y	-xy	xy
x	x	-x	-1	1	xy	-xy	-y	y
-x	-x	x	1	-1	-xy	xy	y	-y
y	y	-y	-xy	xy	-1	1	x	-x
-y	-y	y	xy	-xy	1	-1	-x	x
xy	xy	-xy	y	-y	-x	x	-1	1
-xy	-xy	xy	-y	y	x	-x	1	-1

clearly, 1 is the identity and $-1\cdot g = -g$ $\forall g \in Q_8$. Finally, by properties of $GL_2($ in particular associativity, we know that: $x(xy) = (xx)y = -1y = -y$, and so on.

It remains only to show that for all elements $g \in G$, $g \neq 1, g \neq -1$: $g^2 = -1$.

$$xx = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} i^2 & 0 \\ 0 & (-i)^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1.$$

$$yy = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1.$$

$$zz = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i^2 & 0 \\ 0 & i^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1, \quad \text{AND}$$

$$(-1)(-1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (1)(1)$$

Note also that $yx = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = -xy$

(b) Find all the subgroups of Q_8 and prove that every subgroup is normal

__Solution__: Since $|Q_8| = 8$, by Lagrange's theorem the only possibilities for the size of subgroups of Q_8 are 1, 2, 4, and 8. Subgroups of size 1 and 8 are the trivial subgroup $\langle 1 \rangle$ and Q_8 respectively. It remains to explore possibilities for subgroups of size 2 and 4.

__Subgroups of size 2__: By definition all subgroups must contain the identity. therefore, the only subgroup of size 2 is $\{1,-1\}$, since any other element would have to contain its inverse (g, g^{-1}), which would be more than 2 elem $\{1, g, g^{-1}\}$. Since $\langle -1 \rangle = \{1,-1\}$ and $-1 \in Z(Q_8)$, then $\langle -1 \rangle$ is normal.

__Subgroups of size 4__: From the group table is easy to see that $\{1,-1,x,-x\}$ is a subgroup (look at the first four columns and rows). In fact this is a cyclic subgroup: $\langle x \rangle = \{1, x, x^2, x^3\} = \{1, x, -1, -x\}$. Other cyclic subgroup are: $\langle y \rangle = \{1, y, y^2, y^3\} = \{1, y, -1, -y\}$ and $\langle xy \rangle = \{1, xy, (xy)^2, (xy)^3\} = \{1, xy, -1, -xy\}$ these are all possible groups of size four, since if you try to build any other subgroup, $\{1, x, -x, y, -y\}$, it will have to have the identity, then some other element say x and its inverse $-x$, but then another element say y would have to include its inverse $-y$, which makes 5 elements which can't be a subg

Moreover, all of these are normal. Let us check that $\langle x \rangle \trianglelefteq Q_8$,

Let $g \in Q_8$ then $g\langle x \rangle g^{-1} = \langle x \rangle$. If $g = 1$ or $g = -1$ or $g = x$ or $g = -x$ then t is trivial. Other cases: $y\langle x \rangle y^{-1} = \{y(1)y^{-1}, y(x)y^{-1}, y(-1)y^{-1}, y(-x)y^{-1}\} = \{yy^{-1}, (yx)y^{-1}, -yy^{-1},$
$= \{1, -x(yy^{-1}), -1, x(yy^{-1})\} = \{1, -x, -1, x\} = \langle x \rangle$. From this case follows that $(-y)\langle x \rangle(-y)^{-1} = \langle$

Finally, $(xy)\langle x \rangle(xy)^{-1} = \{(xy)1(xy)^{-1}, (xy)x(xy)^{-1}, (xy)(-1)(xy)^{-1}, (xy)(-x)(xy)^{-1}\} = \{1, x(-xy)y^{-1}x^{-1}, -1, -(xy$
$= \{1, -x, -1, x\} = \langle x \rangle$, therefore $\langle x \rangle$ is normal. (this last case shows $(-xy)\langle x \rangle(-xy)^{-1} = \langle$

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A similar argument shows that $\langle y\rangle$ and $\langle xy\rangle$ are normal. I will ommit the details in interest of time/space.

therefore, Q_8 has 6 subgroups: $\langle 1\rangle, \langle -1\rangle, \langle x\rangle, \langle y\rangle, \langle xy\rangle, Q_8$, all normal.

(c) Find $Z(Q_8)$ and identify the group $Q_8/Z(Q_8)$.

Solution: clearly, $1,-1 \in Z(Q_8)$. this follows from properties of matrix multiplic by scalars: Let $g \in Q_8$, then $-g = (-1)g = g(-1) = -g$. these two elements are the only elements in the center: $x \notin Z(Q_8)$ since $xy \neq -xy = yx$; which also shows that $y \notin Z(Q_8)$. Next, $xy \notin Z(Q_8)$ since $(xy)y = x(yy) = -x \neq x = -x(yy) = (yx)y = y(xy)$. Also, $-x \notin Z(Q_8)$ since $(-x)y \neq xy = y(-x)$. $-y \notin Z(Q_8)$: $x(-y) = -xy \neq xy = (-y)x$; Finally $-xy \notin Z(Q_8)$: $(-xy)y = x \neq y(-xy)$.

therefore $Z(Q_8) = \{1,-1\}$. By definition $Q_8/Z(Q_8) = \{gZ(Q_8) : g \in Q_8\}$.

$= \{1Z(Q_8) = -1Z(Q_8), xZ(Q_8) = -xZ(Q_8), yZ(Q_8) = -yZ(Q_8), xyZ(Q_8) = -xyZ(Q_8)\}$.

$Q_8/Z(Q_8) = \{\{1,-1\},\{x,-x\},\{y,-y\},\{xy,-xy\}\}$

this is the Klein four-group as evidenced by its group table:

	$\{1,-1\}$	$\{x,-x\}$	$\{y,-y\}$	$\{xy,-xy\}$
$\{1,-1\}$	$\{1,-1\}$	$\{x,-x\}$	$\{y,-y\}$	$\{xy,-xy\}$
$\{x,-x\}$	$\{x,-x\}$	$\{1,-1\}$	$\{xy,-xy\}$	$\{y,-y\}$
$\{y,-y\}$	$\{y,-y\}$	$\{xy,-xy\}$	$\{1,-1\}$	$\{x,-x\}$
$\{xy,-xy\}$	$\{xy,-xy\}$	$\{y,-y\}$	$\{x,-x\}$	$\{1,-1\}$

this is even more obvious if we only take representatives of each set in Q

	1	x	y	xy
1	1	x	y	xy
x	x	1	xy	y
y	y	xy	1	x
xy	xy	y	x	1

here x is a representative of $\{x,-$
y is a representative of $\{y,-$
xy is a representative of $\{xy,-$
1 is a representative of $\{1,-$

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(2) Let G be a group and let N be a normal group. Let $\pi: G \to G/N$ denote t canonical homomorphism. Recall that we have shown that if H is any sub group of G then HN is also a subgroup.

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(i) Prove that if H is a subgroup of G then $\pi(H) = \pi(HN)$.

Pf: Let $H \leq G$. First note that for any $n \in N$, $nN = N$, since N is a subgroup so it is closed under the group operation.

($\subseteq$) Let $X \in \pi(H)$. Note that X is a set. In fact $X = hN$, for some $h \in H$. But by previous observation, $N = nN$ for some $n \in N$. Hence, $X = hN = h(nN) = (hn)N$, for some $h \in H$ and $n \in N$. therefore $X = aN$, $a = hn \in HN$, and so $X \in \pi(HN)$. the other direction is very similar:

($\supseteq$) Let $X \in \pi(HN)$. then $X = (hn)N$ for some $hn \in HN$. But then $X = h(nN)$ by associativity and by previous observation $nN = N$, therefore $X = h(nN) = hN$, where $h \in H$, and so $X \in \pi(H)$.

($\subseteq$) and ($\supseteq$) prove that $\pi(H) = \pi(HN)$.

(ii) Prove that if $H \leq G$, $K \leq G$, then $\pi(H) = \pi(K) \Leftrightarrow HN = KN$.

Pf: Let $H \leq G$ and $K \leq G$.

($\Rightarrow$) Suppose $\pi(H) = \pi(K)$.

($\subseteq$) Let $x \in HN$. then $x = hn$ for some $h \in H, n \in N$. But $\pi(H) = \pi(K) = \{hN \mid h \in H\} = \{kN \mid k \in K\}$. therefore, $x = hn \in \{hN\} = \{kN \mid k \in K\}$, which means th there exists $k \in K$ so that $x = kn$, for some $n \in N$. Hence, $x \in KN$.

($\supseteq$) Let $x \in KN$. then $y = kn$ for some $k \in K, n \in N$. But $\pi(H) = \pi(K)$, so following a very similar argument as before, $x = kn \in \{kN\} = \{hN \mid h \in H\}$, so there exists $h \in H$ so that $x = hn$, for some $n \in N$. Hence, $x \in HN$.

($\subseteq$) and ($\supseteq$) prove that $HN = KN$.

($\Leftarrow$) Suppose $HN = KN$

($\subseteq$) let $X \in \pi(H)$. then $X = hN$ for some $h \in H$. As we observed before, pick an element $n \in N$. then $nN = N$. Hence, $X = (hn)N$. But $hn \in HN = KN \Rightarrow hn = kn'$, for some $k \in K$ and $n' \in N$. then, $X = (hn)N = (kn')N = k(n'N) = kN \Rightarrow X \in \pi(K)$. Likewise,

($\supseteq$) Let $X \in \pi(K)$. then $X = kN$ for some $k \in K$. By a very similar argument $X = kN = (kn)N$, but $kn \in KN = HN \Rightarrow kn = hn'$, so $X = (hn')N = hN \Rightarrow X \in \pi(H)$.

($\subseteq$) and ($\supseteq$) prove that $\pi(H) = \pi(K)$.

3. (a). Let G be a group and let x, y be distinct elements in G of order 2. Prove that if x and y commute then $\{e, x, y, xy\}$ is a subgroup of G isomorphic to $C_2 \times C_2$.

Pf: First, let us show that indeed $\{e, x, y, xy\}$ is a subgroup of G.

(i) Each element has an inverse: $e^{-1} = e$. $x^{-1} = x$ since x is of order 2, $x^2 = xx = e$. Likewise, $y^{-1} = y$. Finally $(xy)^{-1} = xy$ because $(xy)(xy) = (xx)(yy)$ by commutativity of x and y, but then $(xy)(xy) = (xx)(yy) = ee = e$.

(ii) the set is closed under the group operation. the only non trivial cases we need to check are: $x(xy) = (xx)y = y$; $(xy)x = x(yx) = (xx)y = y$; $y(xy) = (yx)y = x(yy) = x$; $(xy)y = x(yy) = x$. So G is closed.

Moreover, we have shown that G is abelian. Hence, by theorem proved in class $G \cong C_{n_1} \times C_{n_2} \times \cdots \times C_{n_k}$, for a number k of cyclic subgroups. In this case we can build the isomorphism we want as follows:

Note: $C_2 \cong \langle x \rangle$ and $C_2 \cong \langle y \rangle$, where $\langle x \rangle = \{e, x\}$, $\langle y \rangle = \{e, y\}$. and $\langle x \rangle \times \langle y \rangle = \{(e,e), (e,y), (x,e), (x,y)\}$. Let $f: \langle x \rangle \times \langle y \rangle \to \{e, x, y, xy\}$ be given by $f(a,b) = a \cdot b$. this is clearly a 1-1, onto mapping, Moreover f is a homomorphism: $f((a_1,b_1)\cdot(a_2,b_2)) = f((a_1a_2, b_1b_2)) = (a_1a_2)(b_1b_2) =$ by commutativity $= (a_1b_1)(a_2b_2) = f(a_1,b_1)f(a_2b_2)$ this shows that $G \cong C_2 \times C_2$.

(b) Let G be a finite abelian group of order 8. Prove that G is isomorphic to one of the following 3 groups: C_8, $C_4 \times C_2$ or $C_2 \times C_2 \times C_2$.

Pf: Let G be an abelian group of order 8. By theorem proved in class, we know that an abelian, finite group is isomorphic to the direct product of cyclic subgroups. Pick $g \in G$, $g \neq e$. Look at $\langle g \rangle$. By lagrange's theorem $|\langle g \rangle| = 1, 2, 4$ or 8. It cannot be 1 since $g \neq e$. If $|\langle g \rangle| = 8$, then $\langle g \rangle = G$. So G is a cyclic group of order 8, clearly $G \cong C_8$. otherwise, $|\langle g \rangle| = 2$ or 4. If $|\langle g \rangle| = 4$, then $\langle g \rangle \cong C_4$, and by the previous mentioned theorem $G \cong C_4 \times C_2$; since C_2 is the only other subgroup s.t. $|G| = 8 = |C_4 \times C_2| = 8$.

Finally, if $|\langle g \rangle| = 2$, and there are no other cyclic subgroups of order 4, then by previous mentioned theorem $G \cong C_2 \times C_2 \times C_2$. these are all possibilities.

4. (a) Let $N \trianglelefteq G$. Prove that the one-to-one correspondence π between the subgroups of G that contain N and all of the subgroups of G/N preserve normal subgroups, that is:

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If $K \leq G$ and $N \subseteq K$ then $K \trianglelefteq G \Leftrightarrow \pi(K) \trianglelefteq G/N$.

Pf: Let $K \leq G$ and $N \subseteq K$.

($\Rightarrow$) Suppose $K \trianglelefteq G$. Let $X \in G/N$ and let $Y \in \pi(K)$. then, But then, $X = gN$ for some $g \in G$ and $Y = kN$, for some $k \in K$.

$$X \circ Y \circ X = gN \circ kN \circ g^{-1}N$$
$$= (gN \circ kN) \circ g^{-1}N \quad \text{by associativity}$$
$$= gkN \circ g^{-1}N \quad \text{by definition of } N$$
$$= gkg^{-1}N \quad \text{and therefore,}$$

But by hypothesis K is normal in G, so $gkg^{-1} \in K$ and therefore, $(\underbrace{gkg^{-1}}_{\in K})N \in \pi(K)$. therefore, $\pi(K) \trianglelefteq G/N$.

($\Leftarrow$) Suppose $\pi(K) \trianglelefteq G/N$. then, for any $X \in G/N$ and any $Y \in \pi(K)$, we ha[ve] $X \circ Y \circ X^{-1} \in \pi(K)$. But $X = gN$ for some $g \in G$ and $Y = kN$ for some $k \in K$.

$gN \circ kN \circ g^{-1}N \in \pi(K) \Rightarrow gkg^{-1}N \in \pi(K) \Rightarrow gkg^{-1} \in K$ by definition of $\pi(K)$.

this holds for any $g \in G$ and for any $k \in K$. therefore $K \trianglelefteq G$.

(b) Prove that every finite group G has a homomorphic image that is a sim[ple] group, that is, a nontrivial group with no normal subgroups other than $\{e\}$ and itself.

Pf: By induction on the size of the group. But first note that if G is a sim[ple] group, then take $f: G \to G$ to be the identity. f is an isomorphism and so the homomorphic image of G, i.e., G itself will give the result. therefore, Suppose G is neither trivial nor simple and $|G| > 1$. So G has a nontrivial proper normal subgroup, call it N. We proved in class that $|G/N| = \frac{|G|}{|N|} < |G|$; Hence, we can apply the inductive hypothesis to G/N.

So, there exists a nontrivial simple subgroup $H \leq G/N$, so we have the map $\varphi: G/N \to H$, where φ is an onto homomorphism.

So we have the following diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/N \arrow[ur, dotted, "\varphi"'] & \end{tikzcd}$$

, where as usual π is the cannonical map $\pi(g) = gN$. By theorem proved in class, consider the composition $\pi \circ \varphi$, this is an onto map. Moreover, $f = \varphi \circ \pi$, so f is an onto map.

therefore, $f(G) = (\varphi \circ \pi)(G)$ is a homomorphic image of G. By construction and inductive hypothesis H is simple. since $f(G) = (\varphi \circ \pi)(G) = H$, we have found for any finite group G a homomorphic image that is a simple gro

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(11.8) Let G, G' and H be groups. Establish a bijective correspondence between homomorphisms $\Phi: H \to G \times G'$ from H to the product group and pairs (φ, φ') consisting of a homomorphism $\varphi: H \to G$ and a homomorphism $\varphi': H \to G'$.

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<u>Solution</u>: Let $S = \{\Phi \mid \Phi: H \to G \times G',$ a homomorphism$\}$ and let
$T = \{(\varphi, \varphi') \mid \varphi: H \to G$ a homomorphism and $\varphi': H \to G'$ a homomorphism$\}$

Define $f: S \to T$ by $f(\Phi) = (\varphi, \varphi')$, where $\varphi = \pi_1 \circ \Phi$ and $\varphi' = \pi_2 \circ \Phi$.
Here π_1 and π_2 are the projection maps into G and G' respectively.

As a diagram:
$$\begin{tikzcd} & & G \\ H \arrow[r, "\Phi"] & G \times G' \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & & G' \end{tikzcd}$$

<u>claim</u>: f is 1-1 and onto.

<u>1-1</u>: Let $\Phi_1, \Phi_2 \in S$. suppose that $f(\Phi_1) = f(\Phi_2)$. We want to sho that $\Phi_1 \overset{?}{=} \Phi_2$. Let $h \in H$. then,

$$f(\Phi_1) = f(\Phi_2) \iff (\varphi_1, \varphi_1') = (\varphi_2, \varphi_2') \quad \text{by definition o}$$
$$\iff (\pi_1 \circ \Phi_1, \pi_2 \circ \Phi_1) = (\pi_1 \circ \Phi_2, \pi_2 \circ \Phi_2) \quad \text{by definition of}$$
$$\iff \pi_1 \circ \Phi_1 = \pi_1 \circ \Phi_2 \text{ and } \pi_2 \circ \Phi_1 = \pi_2 \circ \Phi_2$$

Now, consider $(\pi_1 \circ \Phi_1)(h) = \pi_1(\Phi_1(h)) = \varphi_1(h) = \varphi_2(h) = \pi_1(\Phi_2(h)) = (\pi_1 \circ \Phi_2)(h)$.
Likewise $(\pi_2 \circ \Phi_1)(h) = \pi_2(\Phi_1(h)) = \varphi_1'(h) = \varphi_2'(h) = \pi_2(\Phi_2(h)) = (\pi_2 \circ \Phi_2)(h)$.

$\Rightarrow \varphi_1(h) = \varphi_2(h)$ and $\varphi_1'(h) = \varphi_2'(h)$, for any $h \in H$. therefore,
$\Phi_1(h) = (\varphi_1(h), \varphi_1'(h)) = (\varphi_2(h), \varphi_2'(h)) = \Phi_2(h) \Rightarrow \Phi_1 = \Phi_2$.

the function is 1-1.

Onto: Let $(\varphi, \varphi') \in T$. Take $\Phi \in S$ be such that $\varphi = \pi_1 \circ \Phi$ and $\varphi' = \pi_2 \circ \Phi$.
then clearly $f(\Phi) = (\varphi, \varphi')$, and the function is onto

Perhaps an easier way to show that f is a bijection would be to defin
$f^{-1}: T \to S$ given by $f^{-1}(\varphi, \varphi') = \Phi$, where $\Phi(h) = (\varphi(h), \varphi'(h))$; and shou
that f^{-1} is the inverse of f: Let $\Phi \in S$, let $(\varphi, \varphi') \in T$. then:

$$f^{-1} \circ f(\Phi) = f^{-1}(f(\Phi)) = f^{-1}(\varphi, \varphi') = \Phi \text{, so } f^{-1} \text{ is a left inverse}$$
$$f \circ f^{-1}(\varphi, \varphi') = f(f^{-1}(\varphi, \varphi')) = f(\Phi) = (\varphi, \varphi') \text{, so } f^{-1} \text{ is a right inverse.}$$

In any case we have show that f is a bijection from S to T.

(11.9) Let H and K be subgroups of a group G. +10
Prove that HK is a subgroup of G if and only if HK = KH.

($\Rightarrow$) Suppose that HK is a subgroup of G.
($\subseteq$) Let $x \in HK$. then $x = hk$, for some $h \in H$ and some $k \in K$.
But HK is a subgroup so $x^{-1} \in HK$, which means $x^{-1} = h_1 k_1$ for som
$h_1 \in H$, $k_1 \in K$. But then $x = (x^{-1})^{-1} = (h_1 k_1)^{-1} = k_1^{-1} h_1^{-1} \in KH$ since $k_1 \in K \Rightarrow k_1^{-1}$
and $h_1 \in H \Rightarrow h_1^{-1} \in H$. Hence $x \in KH$.
($\supseteq$) Similarly, Let $x \in KH$. then $x = kh$, for some $k \in K$ and some $h \in H$
But HK is a subgroup: $h^{-1} k^{-1} \in HK \Rightarrow (h^{-1} k^{-1})^{-1} = kh \in HK$, therefore $x = kh \in H$

($\Leftarrow$) Suppose that $HK = KH$.

(i) Closed under group operation: Let $x, y \in HK$. By hypothesis $x, y \in KH$
$x = h_1 k_1$, $y = h_2 k_2$ for some $h_1, h_2 \in H$, $k_1, k_2 \in K$. then,

$$xy = (h_1 k_1)(h_2 k_2) = h_1(k_1 h_2) k_2 \text{ ; but } k_1 h_2 \in KH \Rightarrow k_1 h_2 \in HK \Rightarrow k_1 h_2 = h_3$$
for some $h_3 \in$
$k_3 \in K$
$$= h_1 (h_3 k_3) k_2 = (h_1 h_3)(k_3 k_2),$$

and since H and K are subgroups, $h_1 h_3 \in H$, $k_3 k_2 \in K \Rightarrow xy \in HK$.

(ii) Closed under taking inverses: Let $x \in HK$. then $x = hk$, for some $h \in H$ an
some $k \in K$. But then, $x^{-1} = (hk)^{-1} = k^{-1} h^{-1} \in KH$, But $HK = KH$, which means t
$x^{-1} = k^{-1} h^{-1} \in HK$.

(i) and (ii) $\Rightarrow$ HK is a subgroup of G.

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(12.2) In the general linear group $GL_3(\mathbb{R})$, consider the subsets

$$H = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } K = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } * \in \mathbb{R}$$

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(a) SHOW that $H \leq GL_3(\mathbb{R})$.

Pf: (i) closed under group operation: Let $H_1, H_2 \in H$. be like

$H_1 = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, $H_2 = \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}$, where $a, b, c, d, e, f \in \mathbb{R}$. then,

$$H_1H_2 = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d+a & e+af+b \\ 0 & 1 & f+c \\ 0 & 0 & 1 \end{bmatrix}$$

, since $\mathbb{R}$ is closed under addition, we get that

$H_1H_2 = \begin{bmatrix} 1 & g & h \\ 0 & 1 & j \\ 0 & 0 & 1 \end{bmatrix}$, where $g = d+a \in \mathbb{R}$, $h = e+af+b \in \mathbb{R}$ and $j = f+c \in \mathbb{R}$.

$\Rightarrow H_1H_2 \in H$.

(ii) Closed under taking inverses: Let $H_1 \in H$. be $H = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, $a, b, c \in \mathbb{R}$.

then $H^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$, since

$$HH^{-1} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix},$$

Moreover, H^{-1} is of the form $\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$. Hence $H^{-1} \in H$.

(i) and (ii) $\Rightarrow$ H is a subgroup of $GL_3(\mathbb{R})$.

(b) SHOW that $K \trianglelefteq H$.

Let $A \in H$ and $B \in K$. Want to show: $ABA^{-1} \overset{?}{\in} K$.; let $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

we already computed A^{-1} to be $A^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$. Take the product:

$$ABA^{-1} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & (-a) & (ac-b+d) \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in K$$

therefore, K is normal.

ⓒ Identify the quotient group H/K.

By definition $H/K = \{hK \mid h \in H\}$; Let $h = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in H$. then

$hK = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a & *+b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$; So $H/K = \{$ upper triangular matrices with all 1's in its diagonal

ⓓ Determine the center of H.

By definition $Z(H) = \{A \in H \mid AB = BA, \forall B \in H\}$.

An element of the center of H is of the form: $\begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}$, $a, b \in \mathbb{R}$

Since:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & *+a & *+a*+b \\ 0 & 1 & *+a \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+* & b+a*+* \\ 0 & 1 & a+* \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}$$, so $Z(H) = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \middle| \ a, b \in \mathbb{R} \right\}$ //