M403 - Fall 2013 - HW8 - Enrique Areyan

~~8~~ 48/50

1

~~(1)~~ Compute $|Gl_n(\mathbb{F}_p)|$.

Solution: Consider the matrix: $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$; where $a_{ij} \in \mathbb{F}_p$.

to make this an invertible matrix, there are $p^n - 1$ choices for the first column (take away the zero vector). Now, for the second choice we have to exclude from the total number of vectors those that are multiples of the first. So we have $p^n - p$. Likewise, the third vector cannot be multiple of either the first or second so there are $p^n - p^2$. Extending this argument we get:

$$\boxed{|Gl_n(\mathbb{F}_p)| = (p^n-1)(p^n-p)(p^n-p^2)\cdots(p^n-p^{n-1})}$$

For example: $|Gl_2(\mathbb{F}_p)| = (p^2-1)(p^2-p)$. ; $|Gl_3(\mathbb{F}_p)| = (p^3-1)(p^3-p)(p^3-p^2)$.

~~(2)~~ Let p be prime and let R_p denote the following set of matrices:

$$R_p = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{F}_p \right\}.$$

10

(a) Prove that R_p is a commutative ring.

Pf: Need to check the following:

(0) R_p is closed under $+$:
Let $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \in R_p$. then $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} a+c & -(b+d) \\ b+d & a+c \end{pmatrix} \in$

R_p is closed under $\cdot$:
Let $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \in R_p$. then $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}$

(1) $(R,+)$ is an abelian group since:
(i) Clearly $+$ is associative because $+$ is associative in $(\mathbb{F}_p,+)$.
(ii) the identity w.r.t. $+$ is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -0 \\ 0 & 0 \end{pmatrix} \in R_p$.
(iii) Given $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in R_p$ its inverse is $\begin{pmatrix} -a & b \\ -b & -a \end{pmatrix} \in R_p$.

(2) $\cdot$ is associative (this is inherited from the fact that matrix multiplication associative).

(3) Consider $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -0 \\ 0 & 1 \end{pmatrix} \in R_p$. Clearly this is the multiplicative identity
Let $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in R_p$. then $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

(4) Need to show: $\forall A,B,C \in R_p$: $A\cdot(B+C) \overset{?}{=} AB+AC$
$(A+B)\cdot C \overset{?}{=} AC+BC$

M403 - Fall 2013 - <u>HW8</u> - Enrique Areyan.

Let $A=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $B=\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$, $C=\begin{pmatrix} e & -f \\ f & e \end{pmatrix}$: $A,B,C \in R_p$.

$$A\cdot(B+C)=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\cdot\left[\begin{pmatrix} c & -d \\ d & c \end{pmatrix}+\begin{pmatrix} e & -f \\ f & e \end{pmatrix}\right]=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\cdot\begin{pmatrix} c+e & -(d+f) \\ d+f & c+e \end{pmatrix}=$$

$$=\begin{pmatrix} a\cdot(c+e)-b(d+f) & -a(d+f)-b(c+e) \\ a(d+f)+b(c+e) & a(c+e)-b(d+f) \end{pmatrix}=\begin{pmatrix} ac+ae-bd-bf & -ad-af-bc-be \\ ad+af+bc+be & ac+ae-bd-bf \end{pmatrix}$$

$$=\begin{pmatrix} ac-bd+ae-bf & -ad-bc-af-be \\ ad+bc+af+be & ac-bd+ae-bf \end{pmatrix}=\begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}\begin{pmatrix} ae-bf & -(af+be) \\ af+be & ae-bf \end{pmatrix}$$

$$=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}+\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} e & -f \\ f & e \end{pmatrix}=AB+AC.$$

the other direction, i.e., $(B+C)\cdot A = BA+CA$ follows similarly.

(5) $\bullet$ is commutative. Let $A=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $B=\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$, $A,B\in R_p$.

$$A\cdot B=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}=\begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}=\begin{pmatrix} ca-db & -(da+cb) \\ da+cb & ca-db \end{pmatrix}$$

$$=\begin{pmatrix} c & -d \\ d & c \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}=B\cdot A$$

+10

(1), (2), (3), (4) & (5) $\Rightarrow R_p$ is a commutative Ring.

(b) Prove that R_3 and R_7 are fields, but R_5 is not. try to determin for which primes p the ring R_p is a field.

<u>Pf</u>: By definition, R_p will be a field it every non-zero element has an inverse w.r.t. $\bullet$. that is, every non-zero matrix has an inverse. We know that a matrix is invertible iff determinat is not zero. Hence;

<u>For R_3</u>: Let $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\in R_3\setminus\{0\} \Rightarrow \det\begin{pmatrix} a & -b \\ b & a \end{pmatrix}=a^2+b^2$; where $a\neq 0$ or $b\neq$

Possibilities are: $a=0 \Rightarrow b=1,2$ so $1^2=1\not\equiv 0 \pmod 3$; $2^2=4\not\equiv 0 \pmod 3$.
(similar case when $b=0$).

Now, if $a=1 \Rightarrow b=0,1,2 \Rightarrow 1^2+0^2\not\equiv 0 \pmod 3$; $1^2+1^2=2\not\equiv 0 \pmod 3$; $1^2+2^2=5\not\equiv 0$ (mod
(similar case when $b=1$).

Finally, if $a=2 \Rightarrow b=0,1,2 \Rightarrow 2^2+0^2=4\not\equiv 0 \pmod 3$; $2^2+1^2=5\not\equiv 0 \pmod 3$; $2^2+2^2\not\equiv$
(similar case when $b=2$).

So every non-zero matrix in R_3 is invertible w.r.t. $\bullet$ ($\det \neq 0$)

<u>For R_7</u>: We have a similar argument but need to check more cases.

Let us summarize this information on a table:

M403 - Fall 2013 - HW8 - Enrique Areyan

Addition is associative, so I computed only half the table values and copied for convenience.

a\b	0	1	2	3	4	5	6
0	X	1	4	2	2	4	1
1	1	2	5	3	3	5	2
2	4	5	1	6	6	1	5
3	2	3	6	4	4	6	3
4	2	3	6	4	4	7	3
5	4	5	1	6	7	1	7
6	1	2	5	3	3	7	2

Entries on the table are computed as follow:
$a^2+b^2 \pmod 7$.
None of these are zero, therefore all elements of $R_p\setminus\{0\}$ are invertible w.r.t. $\cdot$.

In general, primes p s.t. $p \nmid a^2+b^2$, where $a,b \in \mathbb{F}_p$ and not both a,b are zero; are going to make R_p a field. Otherwise, if $p \mid a^2+b^2$, (same conditions for a,b as before); then $R_p\setminus\{0\}$ will contain a matrix such that its determinant is congruent with zero mod p and hence, not invertible.

(3) Let V be an F-vector space and let W be a subspace. Prove that there is a one-to-one correspondence between the subspaces of V/W and the subspaces of V that contain W.

Pf: We want to find a correspondence: $\begin{matrix} V \\ \mid \\ W_1 \\ \mid \\ W \end{matrix} \longrightarrow V/W$. Consider:
$W_1 \longrightarrow \pi(W_1)=\{\pi(x) \mid x\in W_1\}$; where π is the cannonical map, i.e.,
$\pi(x)=x+W_1$. We have shown that this is a well defined map. We need to show that $\pi(W_1)\leq V/W$. (i) $0\in V/W$ is s.t. $0=\vec{0}+W$. So, since $W\subseteq W_1$, W_1 is a subspace $0\in W_1 \Rightarrow 0=0+W_1\in\pi(W_1)$.
(ii) Let $W_a, W_b\in\pi(W_1)$. $W_a=a+W_1$, $W_b=b+W_1$, $W_a+W_b=(a+W_1)+(b+W_1)=(a+b)\ldots$
$\Rightarrow W_a+W_b\in\pi(W_1)$. So it is closed under addition. Let $\alpha\in F$, $W_a\in\pi(W_1)$, then
$\alpha W_a=\alpha(a+W_1)=\alpha a+W_1$, So α is closed under scalar multiplication.
(i) & (ii) $\Rightarrow \pi(W_1)\leq V/W$.

+10

Now, to prove the one-to-one correspondence, we could prove that the pr… map is 1-1 and onto, or we could just exhibit an inverse function. So consider
$U\leq V/W \longrightarrow \pi^{-1}(U)=\{x\in V \mid \pi(x)=U\}$. We need to show $\pi^{-1}(U)\leq V$, $W\subseteq\pi^{-1}\ldots$ and we are done. Clearly, if $x\in W$ then $x\in\pi^{-1}(U)$, since $\pi(x)$ is some subgroup o…
Moreover: (i) Since $U\leq V/W$, $0+W\in U \Rightarrow 0\in\pi^{-1}(U)$. (ii) Let $v_1,v_2\in\pi^{-1}(U)$. Th…
$\pi(v_1)=U=\pi(v_2)$; $\pi(v_1+v_2) \overset{\text{we proved } \pi \text{ is a homomorphism.}}{=} \pi(v_1)+\pi(v_2)=U+U=U \Rightarrow v_1+v_2\in\pi^{-1}(U)$. Finall…
Let $\alpha\in F$, $v_1\in\pi^{-1}(U) \Rightarrow \pi(\alpha v_1)=\alpha\pi(v_1)=\alpha U=U \Rightarrow \alpha v_1\in\pi^{-1}(U)$.

M403 - Fall 2013 - HW8 - Enrique Areyan

(4) Let V be an n-dimensional vector space over a field F. Define $A_m = \{ f : V^m \to F \mid f \text{ is multilinear, alternating}\}$. Note A_m is a F-v.s.

(a) Prove that if $m > n$, then $A_m = 0$.

Pf: Let $\{v_1, \ldots, v_n\}$ be a basis for V. Let $f \in A_m$ and $u_1, \ldots, u_m \in V$. Consider $f(u_1, \ldots, u_m) = ?$. Now, since $u_1, \ldots, u_m \in V$; we can write each of these vectors uniquely as linear combinations of elements in the basis. Therefore:

$f(u_1, \ldots, u_m) = f(\sum_{i=1}^{n} \alpha_{1i} v_i, \ldots, \sum_{i=1}^{n} \alpha_{mi} v_i)$; for some scalars α_{ji}; $1 \le j \le m$; $1 \le i$

However, since $m > n$, the set $\{u_1, \ldots, u_m\}$ must be linearly dependent. So we can write at least one u_i as a linear combination of all other us. Without loss of generality, say $u_1 = \sum_{i=2}^{m} \beta_i u_i$. Then:

$f(u_1, \ldots, u_m) = f(\sum_{i=2}^{m} \beta_i u_i, u_2, \ldots, u_m)$ since f is multilinear

$= \beta_2 f(u_2, u_2, \ldots, u_m) + \beta_3 f(u_3, u_2, u_3, \ldots, u_m) + \cdots + \beta_m f(u_m, u_2, \ldots, u_m)$ since f is alternating

$= \beta_2 0 + \beta_3 0 + \cdots + \cancel{\beta_m} \cdot 0$

$= \boxed{0}$

Since the choice of u's was arbitrary, we have that any function $f \in A_m$ is zero always, hence $A_m = 0$.

+10

(b) Prove that if $m \le n$, then the dimension of A_m is $\binom{n}{m}$.

Pf: We want to construct a basis for A_m and find that there are $\binom{n}{m}$ vectors (m.a. functions $f : V^m \to F$) in such basis. As usual, let $\{v_1, \ldots, v_n\}$ be a basis for V. Claim: $\{f_1, f_2, \ldots, f_k\}$, where $k = \binom{n}{m}$ and $f_i : V^m \to F$ are m.a. functions given by $f_i(u_1, \ldots, u_m) = \alpha_i f_i(v_{\alpha_1}, \ldots, v_{\alpha_k})$; where $\{v_{\alpha_1}, \ldots, v_{\alpha_k}\}$ is a choice of $\binom{n}{m}$ v_i's from the basis of V; form a basis for A_m.

Pf: To conclude that $\{f_1, \ldots, f_k\}$ forms a basis it suffices to show that this is a linearly independent and spanning set.

Now, since $m \le n$; we can write each of u_i as linear combination of elements of the basis: $f_i(u_1, \ldots, u_m) = f_i(\sum_{i=1}^{n} \beta_{1i} v_i, \ldots, \sum_{i=1}^{n} \beta_{mi} v_i)$. Now, as shown before, some choice of some of the v_i's will cancel leaving us wit $f_i = \alpha_i f(v_{\alpha_1}, \ldots, v_{\alpha_k})$, $\binom{n}{m}$ vectors in the basis V. exactly which depends of the choice of $u_1, \ldots$ but since these are arbitrary, we get the result.

M403 - Fall 2013 - HW8 - Enrique Areyan

(5) Each of the following is a basis of $\mathbb{F}_7^3$ over the field $\mathbb{F}_7$.

$$B = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}, \quad C = \left\{ \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

+4

(a) Find the change of basis matrix from B to C ($M_{B,C}$), that is, find The matrix P such that $P[v]_B = [v]_C$ for all $v \in \mathbb{F}_7^3$.

Solution: We need to compute $P = ([v_1]_C \; [v_2]_C \; [v_3]_C)$, where $v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$. For that purpose we need to solve the system $\begin{bmatrix} 2 & 4 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = v_i$; so that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [v_i]_C$. Hence, let us invert the matrix $\begin{bmatrix} 2 & 4 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix}$ and use it three times to compute $[v_1]_C$, $[v_2]_C$ and $[v_3]_C$: (note: $-1 \equiv 6 \pmod 7$).

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 3 & 2 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 = 4R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 3 & 2 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - (R_1 + R_2)} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 5 & 3 & 6 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 5 & 3 \end{array}\right.$$

$$\xrightarrow{R_2 = 2R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 5 & 5 & 5 \\ 0 & 0 & 5 & 3 & 6 & 1 \end{array}\right] \xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 2 & 2 \\ 0 & 2 & 0 & 5 & 5 & 5 \\ 0 & 0 & 5 & 3 & 6 & 1 \end{array}\right] \xrightarrow{R_3 = 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 2 & 2 \\ 0 & 2 & 0 & 5 & 5 & 5 \\ 0 & 0 & 1 & 2 & 4 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 = 4R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 2 & 2 \\ 0 & 1 & 0 & 6 & 6 & 6 \\ 0 & 0 & 1 & 2 & 4 & 3 \end{array}\right].$$ We can check that indeed: $\begin{pmatrix} 2 & 4 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 6 & 2 & 2 \\ 6 & 6 & 6 \\ 2 & 4 & 3 \end{pmatrix}$

by: $\begin{pmatrix} 2 & 4 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & 6 \end{pmatrix} \begin{pmatrix} 6 & 2 & 2 \\ 6 & 6 & 6 \\ 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 2 \\ 6 & 6 & 6 \\ 2 & 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 4 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & 6 \end{pmatrix}$. Now we can compute:

$$[v_1]_C = \left[\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right]_C \Rightarrow \begin{bmatrix} 2 & 4 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} [v_1]_C = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow [v_1]_C = \begin{bmatrix} 6 & 2 & 2 \\ 6 & 6 & 6 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \boxed{[v_1]_C = \begin{bmatrix} 6 \\ 6 \\ 1 \end{bmatrix}}$$

$$[v_2]_C = \left[\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}\right]_C \Rightarrow \begin{bmatrix} 2 & 4 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} [v_2]_C = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \Rightarrow [v_2]_C = \begin{bmatrix} 6 & 2 & 2 \\ 6 & 6 & 6 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \boxed{[v_2]_C = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}}$$

$$[v_3]_C = \left[\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}\right]_C \Rightarrow \begin{bmatrix} 2 & 4 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} [v_3]_C = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \Rightarrow [v_3]_C = \begin{bmatrix} 6 & 2 & 2 \\ 6 & 6 & 6 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \Rightarrow \boxed{[v_3]_C = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}}$$

Hence, the change of basis matrix P s.t. $P[v]_B = [v]_C$ for all $v \in \mathbb{F}_7^3$ is:

$$\boxed{P = \begin{bmatrix} 6 & 3 & 5 \\ 6 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}}$$ e.g. $[v_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $P[v_1]_B = \begin{bmatrix} 6 \\ 6 \\ 1 \end{bmatrix} = [v_1]_C$.

M403- Fall 2013- HW8 - Enrique Areyan

(b) Let A be the following matrix over F_7:

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & -2 & 0 \\ 3 & 1 & 1 \end{pmatrix}.$$

Find the matrix L_A with respect to the basis B.

Solution. We want to find $L_A = M_{E,B}(T)$, where $E=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ and $B=\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}0\\2\\3\end{pmatrix},\begin{pmatrix}1\\0\\3\end{pmatrix}\right\}$, that is: $M_{E,B}(T)=\left([T(v_1)]_B\ [T(v_2)]_B\ [T(v_3)]_B\right)$

$=\left(\left[T\begin{pmatrix}1\\0\\0\end{pmatrix}\right]_B \left[T\begin{pmatrix}0\\1\\0\end{pmatrix}\right]_B \left[T\begin{pmatrix}0\\0\\1\end{pmatrix}\right]_B\right)=\left(\begin{bmatrix}1\\2\\3\end{bmatrix}_B \begin{bmatrix}0\\-2\\1\end{bmatrix}_B \begin{bmatrix}-1\\0\\1\end{bmatrix}_B\right)$. Again let us invert

$$\left[\begin{array}{ccc|ccc} 1&0&1&1&0&0\\ -1&2&0&0&1&0\\ 1&3&3&0&0&1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1&0&1&1&0&0\\ -1&2&0&0&1&0\\ 0&5&3&0&1&1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1&0&1&1&0&0\\ 0&2&1&1&1&0\\ 0&5&3&0&1&1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1&0&1&1&0&0\\ 0&2&1&1&1&0\\ 0&1&2&0&3&3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1&0&1&1&0&0\\ 0&1&6&1&5&4\\ 0&1&2&0&3&3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1&0&1&1&0&0\\ 0&1&6&1&5&4\\ 0&0&3&6&5&6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1&0&1&1&0&0\\ 0&1&6&1&5&4\\ 0&0&1&2&4&2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1&0&0&6&3&5\\ 0&1&6&1&5&4\\ 0&0&1&2&4&2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1&0&0&6&3&5\\ 0&1&0&3&2&6\\ 0&0&1&2&4&2 \end{array}\right]$$

One can indeed verify that $\begin{bmatrix}1&0&1\\-1&2&0\\1&3&3\end{bmatrix}^{-1}=\begin{bmatrix}6&3&5\\3&2&6\\2&4&2\end{bmatrix}$, since

$$\begin{bmatrix}1&0&1\\-1&2&0\\1&3&3\end{bmatrix}\begin{bmatrix}6&3&5\\3&2&6\\2&4&2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}6&3&5\\3&2&6\\2&4&2\end{bmatrix}\begin{bmatrix}1&0&1\\-1&2&0\\1&3&3\end{bmatrix}.$$

therefore, $\begin{bmatrix}1\\2\\3\end{bmatrix}_B=\begin{bmatrix}6&3&5\\3&2&6\\2&4&2\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}$

$\begin{bmatrix}0\\-2\\1\end{bmatrix}_B=\begin{bmatrix}6&3&5\\3&2&6\\2&4&2\end{bmatrix}\begin{bmatrix}0\\-2\\1\end{bmatrix}=\begin{bmatrix}6\\2\\1\end{bmatrix} \Rightarrow \boxed{L_A=\begin{bmatrix}6&6&6\\4&2&3\\2&1&0\end{bmatrix}}$

$\begin{bmatrix}-1\\0\\1\end{bmatrix}_B=\begin{bmatrix}6&3&5\\3&2&6\\2&4&2\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix}=\begin{bmatrix}6\\3\\0\end{bmatrix}$

↳ with respect to the basis B.

(c) Use your answer to (a) to find the matrix L_A w.r.t. the basis C.

Solution: By (a) we get that $M_C(L_A)=PM_B(L_A)P^{-1}$. We have P, all we need to do is compute P^{-1}.

$$\left[\begin{array}{ccc|ccc} 6&3&5&1&0&0\\ 6&2&3&0&1&0\\ 1&3&4&0&0&1 \end{array}\right] \xrightarrow{R_2=R_2-R_1} \left[\begin{array}{ccc|ccc} 6&3&5&1&0&0\\ 0&6&5&6&1&0\\ 1&3&4&0&0&1 \end{array}\right] \xrightarrow{R_1=R_1-R_3} \left[\begin{array}{ccc|ccc} 5&0&1&1&0&6\\ 0&6&5&6&1&0\\ 1&3&4&0&0&1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 1 & 1 & 0 & 6 \\ 0 & 6 & 5 & 6 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = 4R_3} \left[\begin{array}{ccc|ccc} 5 & 0 & 1 & 1 & 0 & 6 \\ 0 & 6 & 5 & 6 & 1 & 0 \\ 4 & 5 & 2 & 0 & 0 & 4 \end{array}\right] \xrightarrow{R_1 = R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 2 \\ 0 & 6 & 5 & 6 & 1 & 0 \\ 4 & 5 & 2 & 0 & 0 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - 4R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 2 \\ 0 & 6 & 5 & 6 & 1 & 0 \\ 0 & 4 & 6 & 3 & 0 & 3 \end{array}\right] \xrightarrow{R_2 = R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 2 \\ 0 & 2 & 6 & 3 & 1 & 4 \\ 0 & 4 & 6 & 3 & 0 & 3 \end{array}\right] \xrightarrow{R_2 = R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 2 \\ 0 & 5 & 0 & 0 & 1 & 1 \\ 0 & 4 & 6 & 3 & 0 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 = 3R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 & 3 \\ 0 & 4 & 6 & 3 & 0 & 3 \end{array}\right] \xrightarrow{R_1 = R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 5 & 0 & 6 \\ 0 & 1 & 0 & 0 & 3 & 3 \\ 0 & 4 & 6 & 3 & 0 & 3 \end{array}\right] \xrightarrow{R_3 = R_3 - 4R_2} \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 5 & 0 & 6 \\ 0 & 1 & 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 3 & 2 & 5 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - 5R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 6 & 5 \\ 0 & 1 & 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 3 & 2 & 5 \end{array}\right] \xrightarrow{R_3 = 6R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 6 & 5 \\ 0 & 1 & 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & 4 & 5 & 2 \end{array}\right].$$

Hence, $P^{-1} = \begin{bmatrix} 5 & 6 & 5 \\ 0 & 3 & 3 \\ 4 & 5 & 2 \end{bmatrix}$. Indeed we can check that:

$$PP^{-1} = \begin{bmatrix} 6 & 3 & 5 \\ 6 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 & 5 \\ 0 & 3 & 3 \\ 4 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 5 \\ 0 & 3 & 3 \\ 4 & 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 & 3 & 5 \\ 6 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}.$$

Therefore, the matrix L_A w.r.t. the basis C is

8

$$P M_B(L_A) P^{-1} = \begin{bmatrix} 6 & 3 & 5 \\ 6 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 6 & 6 & 6 \\ 4 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 6 & 5 \\ 0 & 3 & 3 \\ 4 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 & 5 \\ 6 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 & 4 \\ 4 & 3 & 4 \\ 3 & 1 & 6 \end{bmatrix}$$

$$\begin{pmatrix} 2 & 0 & 2 \\ 4 & 4 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & 0 & 3 \\ 5 & 2 & 1 \\ 1 & 6 & 5 \end{bmatrix}} \rightarrow \text{matrix } L_A \text{ w.r.t. basis } C.$$

+10