M403 - Fall 2013 - HW9 - Enrique Areyan

(1) Let V be a finite dimensional F-vector space. A linear Transformation $T: V \to V$ is called idempotent if $T^2 = T$. Prove that if T is an idempotent linear Transformation then There is a basis B of V such that the matrix of T with respect to B has the following form

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$$\begin{pmatrix} I_n & O_{n\times m} \\ O_{m\times n} & O_{m\times m} \end{pmatrix}$$

where I_n is the $n\times n$ identity matrix and $O_{r\times s}$ denotes the $r\times s$ zero matrix.

Pf: First note that the only eigenvalues of T are 0 and 1 because: choose any basis for V. Consider $A = M_B(T)$. Clearly $A^2 = A$. But then: Is true that Let $v \neq 0$ be an eigenvector with eigenvalue λ, i.e., $Av = \lambda v$.

$Av = \lambda v = A^2 v = AAv = A\lambda v = \lambda A v = \lambda(\lambda v) = \lambda^2 v \Rightarrow \lambda v = \lambda^2 v \Rightarrow \lambda v - \lambda^2 v = 0$
$\Rightarrow (\lambda - \lambda^2) v = 0$. But since $\vec{v} \neq 0 \Rightarrow \lambda - \lambda^2 = 0 \Rightarrow \lambda = \lambda^2$.

The only solutions of this equation in any arbitrary field F are $\lambda = 0$ or $\lambda = 1$.

Now, we proved in class that if T has a basis consisting of eigenvectors, then there exists a basis $\mathbb{B}$ of V s.t. $M_B(T)$ is diagonal. therefore, if we can find a basis of eigenvectors for the eigenvalues $\lambda = 0$ and $\lambda = 1$, we will prove what we wanted.

$V_0 = \{v \in V : T(v) = 0v = 0\} = Ker(T)$. Let $\{v_1, \dots, v_m\}$ be a basis for $Ker(T)$.
$V_1 = \{v \in V : T(v) = 1\cdot v = v\} = Img(T)$. Let $\{w_1, \dots, w_n\}$ be a basis for $Im(T)$. So
By the dimension theorem, $\dim(V) = \dim(Img(T)) + \dim(Ker(T)) = n + m$. So the set $\mathbb{B} = (w_1, \dots, w_n, v_1, \dots, v_m)$ is s.t. $\{v_1, \dots, v_m\} \cap \{w_1, \dots, w_n\} = \{0\}$ and any vector in V can be written as linear combinations of $\mathbb{B}$. Hence, $\mathbb{B}$ is a basis. Moreover, the matrix of T with respect to B has the desired form.

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$$\left([T[w_1]]_{\mathbb{B}} \vdots [T[w_2]]_{\mathbb{B}} \vdots \cdots \vdots [T[w_n]]_{\mathbb{B}} \vdots [T[v_1]]_{\mathbb{B}} \vdots [T[v_2]]_{\mathbb{B}} \vdots \cdots \vdots [T[v_m]]_{\mathbb{B}} \right) =$$

$$\begin{array}{c} \begin{matrix} 1 & \cdots & n & n+1 & \cdots & m \end{matrix} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & 1 & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array} = \begin{pmatrix} I_n & O_{n\times m} \\ O_{m\times n} & O_{m\times m} \end{pmatrix}$$

(2) Let V be a finite dimensional F-vector space. A linear transformation $T: V \to V$ is called nilpotent if $T^k = 0$, for some positive integer k.

(a) Prove that if T is a nilpotent linear transformation then there is a vector $v \neq 0$ in V such that $T(v) = 0$.

Pf: Let $T: V \to V$ be nilpotent. If $V = \{0\}$, then the result is trivial. Suppose then $V \neq \{0\}$. Pick $v_1 \in V$ s.t. $v_1 \neq 0$. Look at Tv_1. there are two options: $Tv_1 = 0$, in which case we have found $v_1 \neq 0$ such that $Tv_1 = 0$. Otherwise: $Tv_1 \neq 0$. Let $Tv_1 = v_2$, for some $v_2 \in V$; $v_2 \neq 0$. Now apply T again: $T(Tv_1) = Tv_2 \iff T^2 v_1 = Tv_2$. Again, we have two possibilities: either $Tv_2 = 0$, in which case v_2 is the vector we wanted or $Tv_2 \neq 0$. Let $Tv_2 = v_3$ So that we apply T again: $T(T^2 v_1) = T(Tv_2) \iff T^3 v_1 = Tv_3$. Continue this process $k-1$ times. If at any step between 1 and $k-1$ we found v_i with $1 \leq i \leq k-1$ to be s.t. $T(v_i) = 0$, then we are done. Otherwise we have $T^{k-1} v_1 = v_{k-1}$; where $v_{k-1} \neq 0$. Apply T a final time: $T^k v_1 = Tv_{k-1}$; since T is nilpotent: $T^k v_1 = 0$ and v_1 is the vector we wanted, showing the result.

(b) Prove that if W is a T-invariant subspace of V then both $T|_W$ and the induced linear transformation $\bar{T}$ on V/W are nilpotent.

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Pf: Let W be a T-invariant subspace of V, i.e., $T(W) \subseteq W$. Consider the basis $B_1 = (w_1, \ldots, w_m)$ of W. We can complete this basis into a basis of V $(w_1, \ldots, w_m, w_{m+1}, \ldots, w_n)$, And so we may speak of the matrix of $T|_W$ in the basis B_1: $M_{B_1}(T|_W)$ a square $m \times m$ matrix. Look at powers of $M_{B_1}(T|_W)$: $M_{B_1}(T|_W), M_{B_1}^2(T|_W), M_{B_1}^3(T|_W), \ldots$ Since T is a nilpotent linear transformation, there exists a positive integer k such that $T^k = 0$. This means that for every $v \in V$: $T^k v = 0$. In particular, if we restrict T to w we get the result: $w \in W \Rightarrow w \in V$ and therefore, using the same k as before $Tw = 0 \Rightarrow T|_W^k = 0$ so $T|_W^k$ is nilpotent. A similar argument shows that the induced linear transformation is nilpotent: $\bar{T}(v+W) = T(v) + W$ raised to k: $\bar{T}(v+W)^k = T(v)^k + W = 0 + W = \bar{0}$.

(c) Prove that if T is a nilpotent linear transformation then there is a basis B of V such that the matrix of T with respect to B is strictly upper triangular (that is, all of the entries on the diagonal or below are

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Pf: First note that $\lambda = 0$ is the only eigenvalue of T because: $Tv = \lambda v$, $v \neq 0 \Rightarrow T^k v = T^{k-1}(\lambda v) = \lambda T^{k-1}(v) = 0 \Rightarrow \lambda T^{k-1}(v) = 0$ and $T^{k-1}(v) \neq 0 \Rightarrow \lambda = 0$.

Now, consider a basis for the eigenvalue 0. this is the same as a basis for Ker(T). $B = (v_1, \ldots, v_k)$ we can expand this basis to include a basis for Ker(T^2) $B = (v_1, \ldots, v_k, v_{k+1}, \ldots, v_m)$ where $(v_1, \ldots, v_k)$ are a basis for Ker(T) and $v_{k+1}, \ldots, v_m$ a basis for Ker... Since T is nilpotent, eventually we get $Ker(T^k) = V$. Keep expanding the basis B until we get a basis for V: $B = (v_1, \ldots, v_k, v_{k+1}, \ldots, v_m, v_{m+1}, \ldots, v_n)$. Consider $M_B(T)$. this matrix will be strictly upper triangular since $T(v_1), \ldots, T(v_k) \in Ker(T)$ and $T(v_{k+1}), \ldots, T(v_n)$ are in $Ker(T^p)$ for $1 < p \leq k$ and thus are expressed using only vectors from previous $v_{k+1}, \ldots, v_n$ vectors (some, not all of them), the resulting matrix is strictly upper triangular.

(3). (a) Prove that the function $Tr: M_n(F) \to F$ given by sending A to Tr(A) is a linear transformation.

Pf: We want to show:

(i) $Tr(A+B) \stackrel{?}{=} Tr(A) + Tr(B)$

(ii) $Tr(\alpha A) \stackrel{?}{=} \alpha Tr(A)$.

(i) $Tr(A+B) = Tr((a_{ij}) + (b_{ij})) = Tr((a_{ij} + b_{ij})) = \sum_{i=1}^{n} (a_{ii} + b_{ii})$

$= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = Tr(A) + Tr(B)$.

(ii) $Tr(\alpha A) = Tr(\alpha (a_{ij})) = Tr((\alpha a_{ij})) = \sum_{i=1}^{n} \alpha a_{ii} = \alpha \sum_{i=1}^{n} a_{ii} = \alpha Tr(A)$.

$\Rightarrow$ (i) & (ii) mean that Tr is linear.

(b) Prove that for all $A, B \in M_n(F)$, $Tr(AB) = Tr(BA)$

Pf: $Tr(AB) = Tr((a_{ij})(b_{ij})) = Tr(c_{ij})$; where $C = AB$. We can write the elements of C explicitly: $(c_{ij}) = \sum_{k=1}^{n} a_{ik} b_{kj}$; Hence,

$Tr(AB) = Tr((c_{ij})) = Tr\left(\sum_{k=1}^{n} a_{ik} b_{kj}\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji} a_{ij} = Tr(BA)$.

(c) Let $S: V \to V$ be a l.t. and let B, C be bases of V. Prove that $Tr(M_B(S)) = Tr(M_C(S))$. Give a definition of the trace of a l...

Pf: Let $M_C(S) = P M_B(S) P^{-1}$, where P is the change of basis matrix from the basis B to C. then:

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$Tr(M_C(S)) = Tr(P(M_B(S) P^{-1}))$ grouping $M_B(S)P^{-1}$

$Tr(M_C(S)) = Tr((M_B(S) P^{-1}) P)$ By part (b)

$Tr(M_C(S)) = Tr(M_B(S)(P^{-1}P)) = Tr(M_B(S) I) = Tr(M_B(S)) \Rightarrow \boxed{Tr(M_C(S)) = Tr(...}$

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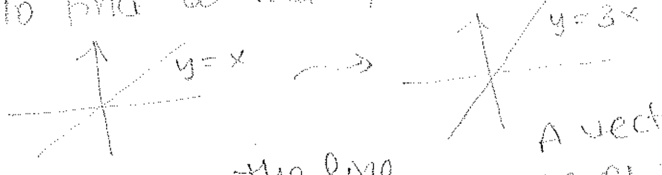
Definition: Let $T:V\to V$ be a linear transformation. Let $\mathcal{B}$ be a arbitrary basis of V. The Trace of T is the trace of the matri[x] of T is the basis $\mathcal{B}$, i.e., $Tr(T)=tr(M_{\mathcal{B}}(T))$. This definition makes sense because the trace function is invariant under choice of basis which we proved before.

(4) From the book, page 126, problem 2.3.
Find all real 2×2 matrices that carry the line $y=x$ to the line $y=3x$.

Solution: we want to find a matrix $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ such that $a,b,c,d\in\mathbb{R}$.

$$A\begin{bmatrix}x_1\\x_1\end{bmatrix}=\begin{bmatrix}x_1\\3x_1\end{bmatrix}$$

$\downarrow$



A vector on the line $y=x$ is of the form: (x_1,x_1); $x_1\in\mathbb{R}$

A vector on the line $y=3x$ is of the form $(x_1,3x_1)$ (or equivalently $(\frac{x_1}{3},x_1)$); $x_1\in\mathbb{R}$.

So, let us solve the linear system:

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}x_1\\x_1\end{bmatrix}=\begin{bmatrix}x_1\\3x_1\end{bmatrix}\Rightarrow\begin{cases}ax_1+bx_1=x_1\\cx_1+dx_1=3x_1\end{cases}$$

$$\Rightarrow\begin{cases}(a+b-1)x_1=0\\(c+d-3)x_1=0\end{cases}$$

If $x_1=0$ then we are free to choose a,b,c,d. Therefore, suppose $x_1\neq 0$.

then $\begin{cases}a+b-1=0\\c+d-3=0\end{cases}$ $\begin{cases}a+b=1\\c+d=3\end{cases}$

therefore, any 2×2 real matrix $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ that satisfies the conditions: (i) $a+b=1$ and (ii) $c+d=3$, will carry the line $y=x$ to the line $y=3x$.

(5) From the book, page 126, problem 3.4.
Let B be a complex $n\times n$ matrix. Prove or disprove: the lin[ear] operator T on the space of all $n\times n$ matrices defined by
$$T(A)=AB-BA$$
is singular.

Solution: claim: the map T has a non trivial kernel.

Pf: $T(I_n)=I_nB-BI_n=B-B=0_{n\times n}$; where I_n is the identity $n\times n$.

So, $I_n \in Ker(T)$. Note that this is not the only matrix in $Ker(T)$.
For example; Let B be an idempotent matrix. Then
$T(B) = BB - BB = B - B = 0$. (this is trivial if $B=0$). (End of claim).
therefore, $Ker(T) \neq \{0\}$ and hence T is not an isomorphism,
which means that T is singular.

(6) From the book, page 128, problem 6.4.
Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Find a matrix P s.t. $P^{-1}AP$ is diagonal, and find
a formula for the matrix A^{30}.

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Solution: First let us find the eigenvalues of A: Eigenvalues of A satisfy

$0 = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{bmatrix}\right) = (\lambda-2)^2 - 1 = \lambda^2 - 4\lambda + 3$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0 \Rightarrow (\lambda-1)(\lambda-3) = 0 \Rightarrow \lambda_1 = 1$ and $\lambda_2 = 3$.

Let us find a basis for the Eigenspaces:

$V_1 = \{v \in \mathbb{R}^2 : Av = v\}$. $\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{cases} 2x_1 + x_2 = x_1 \\ x_1 + 2x_2 = x_2 \end{cases}$

$\Rightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{cases} \Rightarrow x_1 = -x_2 \Rightarrow V_1 = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle$. Hence, an eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $\lambda = 1$

$V_2 = \{v \in \mathbb{R}^2 : Av = 3v\} \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{cases} 2x_1 + x_2 = 3x_1 \\ x_1 + 2x_2 = 3x_2 \end{cases}$

$\Rightarrow \begin{cases} -x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{cases} \Rightarrow x_1 = x_2 \Rightarrow V_2 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$. Hence, an eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $\lambda = 1$.

the matrix P is: $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. the inverse can be computed as follows:

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \xrightarrow{R_2 = \frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right] \xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & -1/2 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right]$. check:

$P^{-1} = 1/2\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow PP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} 1/2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1/2\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. So we can diagonalize A.

$D = P^{-1}AP = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Finally, let us find a formula for A^{30}. Note that $D = P^{-1}AP \Rightarrow$
$D^{30} = (P^{-1}AP)^{30}$; but $(P^{-1}AP)^{30} = P^{-1}A\cancel{(PP^{-1})}A\cancel{(PP^{-1})}\cdots\cancel{PP^{-1}}AP = P^{-1}AA\cdots AP = P^{-1}A^{30}P$

$\Rightarrow D^{30} = P^{-1}A^{30}P \Rightarrow \boxed{A^{30} = PD^{30}P^{-1}}$; where $D^{30} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^{30} = \begin{bmatrix} 1^{30} & 0 \\ 0 & 3^{30} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3^{30} \end{bmatrix}$

concretely: $A^{30} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3^{30} \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = 1/2\begin{bmatrix} 3^{30}+1 \\ 3^{30}-1 \end{bmatrix}$