M403 Homework 10

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1. The following table shows a bijection from the set of rational numbers to the set of natural numbers:

$\mathbb{Q}:$	0	$\frac{1}{1}$	$\frac{1}{2}$	$-\frac{1}{1}$	$\frac{2}{1}$	$-\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{2}{1}$	$\frac{3}{1}$	$\frac{2}{3}$	$-\frac{1}{4}$	...
$\mathbb{N}:$	1	2	3	4	5	6	7	8	9	10	11	12	13	...

This bijection covers all of the rational numbers, which are mapped to one and only one natural number. Hence, $\mathbb{Q}$ is countable.

2. Since X is infinite countable, there exists a bijection $f : \mathbb{N} \to X$. This means that we can lay the elements of X one after the other in the order given by the bijection: $f(0), f(1), f(2), ...$ Now suppose that Y is an infinite set contained in X. We can index the elements of Y as follow: let n_1 be the position of the first element of Y in X as given by f, n_2 the second, n_3 the third and so on. Then we can list the elements of Y as $f(n_1), f(n_2), f(n_3), ...$, which is clearly a bijection from the natural numbers to Y and hence, Y is countable.

3. Proof by contradiction. Suppose that the set of infinite sequences of integers, X, is countable, i.e., there exists a bijection from the natural numbers to this set. Since X is countable, we can list all its elements:

$$(n_1^0, n_2^0, ...), (n_1^1, n_2^1, ...), (n_1^3, n_2^3, ...), ..., (n_1^i, n_2^i, ...), ...$$

This means that $f(0) = (n_1^0, n_2^0, ...), f(1) = (n_1^1, n_2^1, ...), ..., f(i) = (n_1^i, n_2^i, ...), ...$
Consider the following sequence of integers: $(n_i^i + 1)$ for $i = 1, 2, ...$ This is a sequence of integers since every n_i^i is by definition an integer and adding one to an integer yields another integers. This sequence is different from every sequence in our list since it is different in exactly one element and hence, this sequence is not in our list. In other words, there is no $j \in \mathbb{N}$ such that $f(j) = (n_i^i + 1)$. This means that f is not surjective and hence, it is not bijective in contradiction to our assumption. Therefore, there exists no bijection $f : X \to \mathbb{N}$, which means that X is not countable.

(2.8) Let $f : X \to Y$ be a function. Suppose that the function g is the inverse of f. To show that g is a bijection we need to show that (i) g is injective and (ii) g is surjective.

- (i) Let $y_1, y_2 \in Y$ be such that $g(y_1) = g(y_2)$. If we apply f to both sides of this equation we obtain: $f(g(y_1)) = f(g(y_2)) \iff (f \circ g)(y_1) = (f \circ g)(y_2)$ by definition of function composition. Now, since g is the inverse of f, we know that $f \circ g = 1_Y$, and hence $(f \circ g)(y_1) = (f \circ g)(y_2) \iff 1_Y(y_1) = 1_Y(y_2) \iff y_1 = y_2$, which means that g is injective.
- (ii) Let $x \in X$. Apply f to x, i.e., $f(x)$. Call this element y: $f(x) = y$. Now, apply g to both sides of this equation: $g(f(x)) = g(y) \iff (g \circ f)(x) = g(y)$ by definition of function composition. But, since g is the inverse of f, $g \circ f = 1_X$ and so $1_X(x) = g(y) \iff x = g(y)$, which means that g is surjective.

(2.9) Suppose that $f : X \to Y$ is a bijection with two inverses, i.e., there exists two functions $f_1 : Y \to X$ and $f_2 : Y \to X$ such that:

$$f \circ f_1 = 1_Y \text{ and } f_1 \circ f = 1_X$$
$$f \circ f_2 = 1_Y \text{ and } f_2 \circ f = 1_X$$

But then:

$$\begin{aligned} f_1 &= f_1 \circ 1_Y && \text{since } 1_Y \text{ is the identity function} \\ &= f_1 \circ (f \circ f_2) && \text{by definition of } 1_Y \\ &= (f_1 \circ f) \circ f_2 && \text{since function composition is associative} \\ &= 1_X \circ f_2 && \text{by definition of } 1_X \\ &= f_2 && \text{since } 1_X \text{ is the identity function} \end{aligned}$$

Hence, $f_1 = f_2$, which means that the inverse of f is unique.

(2.10) By proposition 2.10, it suffices to find an inverse function g for f which would imply that f is bijective.
Claim: the inverse of f, call it g is $g : \mathbb{R} \to \mathbb{R}$ defined as $g(x) = \frac{x-5}{3}$. Proof:

$$(g \circ f)(x) = g(f(x)) = g(3x + 5) = \frac{3x + 5 - 5}{3} = \frac{3x}{3} = x \Longrightarrow g \circ f = I_{\mathbb{R}}$$

$$(f \circ g)(x) = f(g(x)) = f(\frac{x-5}{3}) = 3(\frac{x-5}{3}) + 5 = x - 5 + 5 = x \Longrightarrow f \circ g = I_{\mathbb{R}}$$

Hence, g is the inverse of f, which also means that f is bijective.

(2.11) f **is not a function**. Consider $(\frac{13}{1}, \frac{1}{2}) = (\frac{13}{1}, \frac{2}{4}) \in \mathbb{Q} \times \mathbb{Q}$. Apply f: $f(\frac{13}{1}, \frac{1}{2}) = \frac{14}{3} \neq 3 = \frac{15}{5} = f(\frac{13}{1}, \frac{2}{4})$. Hence, the same input yields two different outputs, which violates the definition of a function.

(2.16) (i) The proof is by double containment.

($\subseteq$). Let $x \in f(\bigcup_{i \in I} S_i)$. Then there exists $s \in \bigcup_{i \in I} S_i$ such that $f(s) = x$. By definition of union of sets, there exists a set that contains s, i.e., there exists $k \in I$ such that $s \in S_k$, which means that $x = f(s) \in f(S_k)$. But if the element is in one of $f(S_k)$, then it is certainly in the union of all sets S: $x \in \bigcup_{i \in I} f(S_i)$.

($\supseteq$). Let $x \in \bigcup_{i \in I} f(S_i)$. Then there exists $k \in I$ such that $x \in f(S_k)$ which means that there exists $s \in S_k$ for which $f(s) = x$. But if x is in $f(S_k)$, then by definition of union it is in $x \in f(\bigcup_{i \in I} S_i)$.

(ii) Counterexample: Let $X = \{1, 2, 3\}$ and $Y = \{1, 2, 3\}$. Let $f : X \to Y$ be defined as $f(1) = f(2) = 1$ and $f(3) = 3$. Consider the subsets $S_1 = \{2, 3\} \in X$ and $S_2 = \{1\} \in X$. Then, $S_1 \cap S_2 = \emptyset$ and $f(\emptyset) = \emptyset$. Also, $f(S_1) = \{1, 3\}$ and $f(S_2) = \{1\}$ and $f(S_1) \cap f(S_2) = \{1\} \neq \emptyset$.

Now back to the general case. We want to show that $f(S_1 \cap S_2) \overset{?}{\subseteq} f(S_1) \cap f(S_2)$. Let $x \in f(S_1 \cap S_2)$. Then there exists $s \in S_1 \cap S_2$ such that $f(s) = x$. By definition of intersection $s \in S_1 \cap S_2$ means that $s \in S_1$ and $s \in S_2$. This implies that $x = f(s) \in f(S_1)$ and $x = f(s) \in f(S_2)$ which by definition of intersection is the same as $x \in f(S_1) \cap f(S_2)$

(iii) The proof is by double containment.

($\subseteq$) done in part (ii)

($\supseteq$) Let $x \in f(S_1) \cap f(S_2)$. By definition of intersection $x \in f(S_1)$ and $x \in f(S_2)$, which in turn means that there exists $s_1 \in S_1$ and $s_2 \in S_2$ such that $x = f(s_1) = f(s_2)$. Since, f is injective we conclude that $s_1 = s_2$, and thus, $s_1, s_2 \in S_1 \cap S_2 \Rightarrow x \in f(S_1 \cap S_2)$.

(2.17) (i.i) The proof is by double containment.

($\subseteq$). Let $x \in f^{-1}(\bigcup_i B_i)$. Then, by definition of inverse image, $f(x) \in \bigcup_i B_i$, which by definition of union means that $f(x) \in B_k$ for some k. Again, applying definition of inverse image we conclude that $x \in f^{-1}(B_k)$. If x belong to one $f^{-1}(B_k)$ it certainly belong to hte union of all, i.e., $x \in \bigcup_i f^{-1}(B_i)$

($\supseteq$) Let $x \in \bigcup_i f^{-1}(B_i)$. Then there exists k such that $x \in f^{-1}(B_k)$, which means that $f(x) \in B_k$ and it is certainly in the union $x \in f^{-1}(\bigcup_i B_i)$

(i.ii) The proof is by double containment.

($\subseteq$). Let $x \in f^{-1}(\bigcap_i B_i)$. Then, by definition of inverse image, $f(x) \in \bigcap_i B_i$, which by definition of intersection means that $f(x) \in B_k$ for all k. Again, applying definition of inverse image we conclude that $x \in f^{-1}(B_k)$, and since this is true for all k we have that $x \in \bigcap_i f^{-1}(B_i)$.

($\supseteq$) Let $x \in \bigcap_i f^{-1}(B_i)$. Then for all k is true that $x \in f^{-1}(B_k)$, which means that $f(x) \in B_k$. Since this is true for all k it follows that $x \in f^{-1}(\bigcap_i B_i)$

(ii) The proof is by double containment.

($\subseteq$). Let $x \in f^{-1}(B')$ By definition of inverse image, $x \in B' \iff x \in Y - B$ by definition of complementation. This means that $f(x) \notin B \iff x \notin f^{-1}(B) \iff x \in f^{-1}(B)'$
($\supseteq$). Let $x \in f^{-1}(B)'$. By definition of complementation $x \notin f^{-1}(B)$, which means that $f(x) \notin B \iff f(x) \in B'$. By definition of inverse image, this means that $x \in f^{-1}(B')$.