Enrique Areyan - eareyan@umail.iu.edu. M409 (1)

Homework 1:

100/100 very good work.

Section 1.1:

(1) Show that if c is a number, the $c\Theta = \Theta$.

$c\Theta = c(\Theta+\Theta)$ ...... (VS2) Identity element applied in particular to Θ itself, given that $\Theta \in V$.

$c\Theta = c\Theta + c\Theta$ ...... (VS5) $d(u+v) = du + dv$, where d is a number and $u,v \in V$.

$\underbrace{-(c\Theta)+c\Theta} = (-(c\Theta)+c\Theta) + c\Theta$ ... (VS3) add the inverse of the element $c\Theta$ to both sides. Also, (VS1), Associativity.

$\Theta = \Theta + c\Theta$ .... (VS3) an element plus its inverse equal Θ.

$\boxed{\Theta = c\Theta}$ .... (VS2).

(2) c is a number and $c \neq 0$. $v \in V$. Prove if $cv = \Theta$, then $v = \Theta$

$\Theta = v + (-v)$ .... (VS3) inverse of element $v \in V$.

$c\Theta = c(v+(-v))$ .... Operating by c both sides of equation.

$c\Theta = cv + c(-v)$ .... (VS5)

$c\Theta = \Theta + c(-v)$ .... By hypothesis, $cv = \Theta$

$c\Theta = c(-v)$ .... (VS2) Identity element Θ.

$\Theta = c(-v)$ .... Using what wase proved in the previous exercise, i.e., $c\Theta = \Theta$

$c^{-1}\Theta = (c^{-1}c)(-v)$ .... c is an element of the field associated with V. thus, because $c \neq 0$ by hypotlesis, we can operate by its multiplicative inverse.

$\Theta = 1.(-v)$ .... Here, I applied $c\Theta = \Theta$ to the left side and (VS7) to the right. Finally, applying (VS8);

$\Theta = -v$

Now add v to both sides $\Rightarrow$ $v + \Theta = v + (-v) \overset{(VS2 \& VS3)}{\Rightarrow}$

$\boxed{v = \Theta}$

(3) the condition (VS2) (Identity with respect to sum) of the function V.S. is given by: $\theta(x)=0$, $\forall x$ in f's domain.

Proof: let $f(x) \in V$ be a member of the space of functions.

then:

$(f+\theta)(x) = f(x)+\theta(x)$ ...... By definition of sum of functions.

$f(x)+\theta(x) = f(x)+0$ ...... By definition of θ.

$f(x)+0 = f(x)$ ..... By properties of real numbers

$\Rightarrow (f+\theta)(x) = f(x)$ ..... Satisfies (VS2)

Also note that $(\theta+f)(x) = f(x)$, because addition is commutative on the underlying field for this vector space.

(5) $v,w \in V$ and $v+w=v$. Show that $w=\theta$

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$v+w = v$ ..... Hypothesis

$((-v)+v)+w = -v+v$ .... Operating by the inverse of v (VS3) and associating elements (VS1)

$\theta + w = \theta$ ..... Definition of additive identity (VS2)

$\Rightarrow w=\theta$. this shows that the additive identity is unique!

Section 1.2

1(a) $(1,1,1)$ and $(0,1,-2)$. the strategy to show linearly independence is going to be the same from 1(a) to 1-(h). We need to solve the eqs: $X\cdot(1,1,1) + Y\cdot(0,1,-2) = (0,0,0)$. this is the same as solving the following system (Here X and Y are numbers):

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{cases} 1\cdot X + 0\cdot Y = 0 \Rightarrow \boxed{X=0} \\ 1\cdot X + 1\cdot Y = 0 \\ 1\cdot X - 2\cdot Y = 0 \end{cases}$. If I replace this in the next eq. I obtain:

$0+y=0 \Rightarrow \boxed{y=0}$.

therefore, the solution of this system of eqs. is only the trivial $X=y=0$, which implies that $(1,1,1)$ and $(0,1,-2)$ are independent. In what follows, I'm just going to solve the appropiate system without this detailed explanation.

Enrique Areyan - eareyan@umail.iu.edu

Section 1.2

1.(b). $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; $\begin{cases} 1\cdot x + 1y = 0 \Rightarrow x+y=0 \Rightarrow x+0=0 \Rightarrow \boxed{x=0} \\ 0\cdot x + 1\cdot y = 0 \Rightarrow \boxed{y=0} \end{cases}$

the only sol. is $x=y=0$. thus $\{(1,0),(1,1)\}$ is L.I.

1.(c) $\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; $\begin{cases} -x + 0\cdot y = 0 \Rightarrow \boxed{x=0} \\ x+y=0 \\ 0\cdot x + 2\cdot y = 0 \end{cases}$ $\Downarrow$ $0+y=0 \Rightarrow \boxed{y=0}$

the only sol is $x=y=0$. thus $\{(-1,1,0),(0,1,2)\}$ is L.I.

1.(d) $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{cases} 2x+y=0 \\ -x+0y=0 \Rightarrow \boxed{x=0} \end{cases}$ $\Uparrow$ $2(0)+y=0 \Rightarrow \boxed{y=0}$

the only sol is $x=y=0$. thus $\{(2,-1),(1,0)\}$ is L.I.

1(e) $\begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{cases} \pi\cdot x + 0\cdot y = 0 \\ 0\cdot x + 1\cdot y = 0 \Rightarrow \boxed{y=0} \end{cases}$ $\Uparrow$ $\pi x = 0 \Rightarrow \boxed{x=0}$

the only sol is $x=y=0$. thus $\{(\pi,0);(0,1)\}$ is L.I.

1(f) $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{cases} x+y=0 \Rightarrow x=-y \\ 2x+3y=0 \end{cases}$ $\Downarrow$ $-2y+3y=0 \Rightarrow \boxed{y=0}$ $\Uparrow$ $x=-(0) \Rightarrow \boxed{x=0}$

the only sol is $x=y=0$. thus $\{(1,2),(1,3)\}$ is L.I.

1(g) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; $\begin{cases} x+y+0\cdot z = 0 \Rightarrow x+y=0 \Rightarrow x=-y \\ x+y+z=0 \\ 0x+y-z=0 \end{cases}$ $\Downarrow$ $-y+y+z=0 \Rightarrow \boxed{z=0}$ $\Rightarrow y-z=0 \Rightarrow \boxed{y=0}$

Because $x=-y$ and $y=0$, $\boxed{x=0}$

the only solution is the trivial $x=y=z=0$. these are L.I.

1(h) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; $\begin{cases} 0\cdot x + 0\cdot y + z = 0 \Rightarrow \boxed{z=0} \\ 1\cdot x + 2\cdot y + 5\cdot z = 0 \\ 1\cdot x + 1\cdot y + 3\cdot z = 0 \end{cases}$ $\Downarrow$ $x+y=0 \Rightarrow x=-y$

Replacing $x=-y$ and $z=0$ into $x+2y+5z=0 \Leftrightarrow -y+2y+5\cdot 0=0$ $\Rightarrow \boxed{y=0}$. Replacing this into $x=-y \Rightarrow \boxed{x=0}$

Hence, the only solution is the trivial solution $x=y=z=0$. these are L.I.

1.2.2.

(a) $X=(1,0)$, $A=(1,1)$, $B=(0,1)$

Just like exercise 1.2.1, we need to solve a system of eqs. but a little different: $x(1,1)+y(0,1)=(1,0)$, where x, y are numbers.

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\begin{cases} x+0.y=1 \Rightarrow \boxed{x=1} \\ x+y=0 \end{cases}$ $\Downarrow$ $1+y=0 \Rightarrow \boxed{y=-1}$

So, the vector X can be expressed as a linear combination of A and B as follow: $1.(1,1)+(-1).(0,1)=(1,1)+(0,-1)$ $=(1+0,1-1)=(1,0)=X$. the coordinates of X with respect to A, B are $(1,-1)$.

1.2.2 (b). $X=(2,1)$, $A=(1,-1)$, $B=(1,1)$

$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$; $\begin{cases} x+y=2 \Rightarrow x=2-y \\ -x+y=1 \end{cases}$ $\Downarrow$ $y-2+y=1 \Rightarrow 2y=3 \Rightarrow \boxed{y=3/2}$

$x=2-\frac{3}{2} \Rightarrow \boxed{x=1/2}$

the coordinates of X are $(1/2, 3/2)$.

$\frac{1}{2}(1,-1)+\frac{3}{2}(1,1)=(\frac{1}{2},-\frac{1}{2})+(\frac{3}{2},\frac{3}{2})=(\frac{1}{2}+\frac{3}{2}, -\frac{1}{2}+\frac{3}{2})=(2,1)=X$

1.2.2 (c) $X=(1,1)$, $A=(2,1)$, $B=(-1,0)$

$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\begin{cases} 2x-y=1 \\ x+0.y=1 \end{cases} \Rightarrow \boxed{x=1}$

$2(1)-y=1 \Rightarrow 2-1=y \Rightarrow \boxed{y=1}$

the coordinates of X are $(1,1)$

$1.(2,1)+1.(-1,0)=(2-1,1+0)=(1,1)=X$.

1.2.2 (d) $X=(4,3)$, $A=(2,1)$, $B=(-1,0)$

$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 4 \\ 3 \end{bmatrix}$; $\begin{cases} 2x-y=4 \\ x+0.y=3 \end{cases} \Rightarrow \boxed{x=3}$

$6-y=4 \Rightarrow \boxed{2=y}$

the coordinates are $(3,2)$

$x.A+y.B=3(2,1)+2(-1,0)=(6,3)+(-2,0)=(4,3)=X$

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Enrique Areyan - eareyan@umail.iu.edu. M409. Homework 1. ③

Section 1.2

4.- $(a,b),(c,d) \in \mathbb{R}^2$. If $ad-bc=0$, show that they are L.D. If $ad-bc \neq 0$, show that they are L.I.

Solution: (i) Linearly Dependence. Suppose that $ad-bc=0$. TAKE the linear combination

$$e(a,b)+f(c,d)=0$$
$$(ea,eb)+(fc,fd)=0 \Rightarrow \begin{cases} ea+fc=0 & (1) \\ eb+fd=0 & (2) \end{cases}$$

If we multiply the first equation (1) by d and the second (2) by c:

$$\begin{cases} ead+fcd=0 & (3) \\ ebc+fdc=0 & (4) \end{cases}$$

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Now, substract (4) from (3)

$$ead-ebc+\cancel{fcd}-\cancel{fdc}=0 \Rightarrow ead-ebc=0 \Rightarrow \boxed{e(ad-bc)=0}$$

thus, If $ad-bc=0$, e can take any value and thus, not all coefficients are zero and the vectors $(a,b),(c,d)$ are dependent.

(ii) Linearly Independence. Suppose that $ad-bc \neq 0$. through the same procedure as before we obtain $e(ad-bc)=0$. Now, If $ad-bc \neq 0$, we can divide this relation by $ad-bc$

$$\frac{e(ad-bc)}{ad-bc}=\frac{0}{ad-bc} \Rightarrow \boxed{e=0}$$ Replacing this fact in (1) & (2)

$$0 \cdot a + f \cdot c = 0 \Rightarrow fc=0$$
$$0 \cdot b + f \cdot d = 0 \Rightarrow fd=0$$

If either c or d or both are not zero, then $\boxed{f=0}$

Note that if both c and d are zero, then $(c,d)=\Theta$, which is always trivially L.D. of any other vector.

1.2.5. (a) $1, t$.

these pair of functions are independent iff:

$\forall t: \quad a \cdot 1 + b \cdot t = 0 \implies a = b = 0$

we can differentiate the hypotesis:

$\frac{d}{dt}(a + bt) = \frac{d}{dt}(0) \implies \boxed{b=0}$. If we replace this fact In the hypothesis: then,

$a + (0)t = 0 \implies \boxed{a=0}$.

thus, $a=b=0$, the pair is independent.

(b) t, t^2. Again, independence means:

$\forall t: \quad at + bt^2 = 0 \implies a = b = 0$

Same as before, differentiating:

$\frac{d}{dt}(at+bt^2) = \frac{d}{dt}(0) \implies a + 2bt = 0$ (1). Once again:

$\frac{d}{dt}(a+2bt) = \frac{d}{dt}(0) \implies 2b = 0 \implies \boxed{b=0}$. If we replace this back into (1).

$a + 2(0)t = 0 \implies \boxed{a=0}$

thus, $a=b=0$, the pair is independent.

(c) t, t^4. $\forall t: \quad at + bt^4 = 0 \implies a = b = 0$

We can use the same strategy as before. However, we can also prove by reduction to absurd:

Note: the other cases $a=0$ and $b \neq 0$ or $a \neq 0$ and $b=0$ are very similar to this one

Suppose a and b are not zero: then,

$at = -bt^4$, divide by $-bt$

$\frac{a}{-b} = t^3$. But a and b are constant numbers and this equation does not hold for all t. (for instance, pick $t = \pi$).

therefore, our inital assumption is wrong and it must be the case that $a=b=0$, proving that t, t^4 are L.I.

(d) e^t, t $\forall t: ae^t + bt = 0 \Rightarrow a=b=0$

Taking the derivative in both sides of the hypothesis

$\frac{d}{dt}(ae^t + bt) = \frac{d}{dt}(0) \Rightarrow ae^t + b = 0$, $^{(1)}$ differetiating again:

$\frac{d}{dt}(ae^t + b) = \frac{d}{dt}(0) \Rightarrow ae^t = 0$, but $e^t > 0$, so $\boxed{a=0}$

Plugging this back into (1):

$0e^t + b = 0 \Rightarrow \boxed{b=0}$. thus, $a=b=0$, the pair is L.I.

(e) te^t, e^{2t} $\forall t \cdot ate^t + be^{2t} = 0 \Rightarrow a=b=0$

$ate^t + be^{2t} = 0 \Rightarrow e^t(at + be^t) = 0$. but, $e^t > 0$ $\forall t$,

So we can divide by e^t to obtain $at + be^t = 0$ Taking $\frac{d}{dt} \Rightarrow$

$\frac{d}{dt}(at + be^t) = \frac{d}{dt}(0) \Rightarrow \boxed{a + be^t = 0}$. If $a=0$, then $b=0$ because $e^t > 0$ $\forall t$.

On the other hand, If $b=0$ then $a=0$ is trivial. Suppose, however that both a and b are not zero, then

$a + be^t = 0 \Rightarrow be^t = -a \Rightarrow e^t = \frac{-a}{b}$. But this relation does not hold for all t (e.g. $t=1$). therefore, it must be the case that $a=b=0$. the pair te^t, e^{2t} is L.I.

(f) $\sin t, \cos t$ $\forall t \cdot a\sin t + b\cos t = 0 \Rightarrow a=b=0$

TAKE $t=0$, then $a \cdot \sin(0) + b \cdot \cos(0) = 0 \Rightarrow \boxed{b=0}$

But, TAKE $t = \pi/2$, then $a \cdot \sin(\pi/2) + b \cdot \cos(\pi/2) = 0 \Rightarrow \boxed{a=0}$

thus, in order for the hypothesis to hold, both a and b must be zero, i.e, $\sin t$ & $\cos t$ are L.I.

(g) $t, \sin t$ $\quad \forall t: \ at + b\sin t = 0 \Rightarrow a = b = 0$

$$\frac{d}{dt}(at + b\sin t) = \frac{d}{dt}(0) \Rightarrow a + b\cos t = 0$$

this relation must hold for all t: In particular for

$t = \pi/2 \Rightarrow a + b\cos(\pi/2) = 0 \Rightarrow \boxed{a = 0}$

$t = 0 \Rightarrow a + b(\cos(0)) = 0 \Rightarrow a + b = 0 \Rightarrow 0 + b = 0 \Rightarrow \boxed{b = 0}$

these are L.I.

(h) $\sin t, \sin 2t$. $\quad \forall t \quad a\sin t + b\sin(2t) = 0 \Rightarrow a = b = 0$

TAKE $t = \pi/2$ $\quad a \cdot \sin(\pi/2) + b\sin(\pi) = 0 \Rightarrow a + b.0 = 0 \Rightarrow \boxed{a = 0}$

the hypothesis has to hold for all values of t, in particular $t = \pi/2$, thus we can conclude that $a = 0$ and replace this in the original eq:

$$0 \cdot \sin t + b \cdot \sin(2t) = 0 \Rightarrow b \cdot \sin(2t) = 0$$

TAKE $t = \pi/4 \Rightarrow b \cdot \sin(\pi/2) = 0 \Rightarrow \boxed{b = 0}$

these are L.I.

(i) $\cos t, \cos 3t$. $\quad \forall t: \ a\cos t + b\cos 3t) = 0 \Rightarrow a = b = 0$

this relation must hold for all values of t, In particular:

TAKE $t = 0 \Rightarrow a\cos(0) + b\cos(0) = 0$

$$\Rightarrow a + b = 0 \quad \Rightarrow a = -b.$$

TAKE $t = \pi/6 \Rightarrow a\cos(\pi/6) + b\cos(\pi/2) = 0$

$$a\left(\frac{\sqrt{3}}{2}\right) + b \cdot 0 = 0 \quad \Rightarrow a\frac{\sqrt{3}}{2} = 0$$

$$\Rightarrow \boxed{a = 0}$$

If $a = 0$, and $a = -b \Rightarrow \boxed{b = 0}$

these are L.I.

1.2.6 (a).

$t > 0$. (a) $t, 1/t$

$\forall t : t > 0 : \quad at + \frac{b}{t} = 0$

TAKE $t=1$: $a + b = 0 \Rightarrow a = -b$.

TAKE $t=2$: $2a + \frac{1}{2}b = 0 \Rightarrow -2b + \frac{1}{2}b = 0$

$b(-2 + \frac{1}{2}) = 0 \Rightarrow b(-\frac{3}{2}) = 0 \Rightarrow \boxed{b=0}$

If $b=0$ and $a=-b \Rightarrow \boxed{a=0}$

these are L.I.

1.2.6 (b)

$t > 0$. $e^t, \log t$

$\forall t : t > 0 : \quad ae^t + b\log(t) = 0$

TAKE $t=1$: $ae^1 + b\log(1) = 0 \Rightarrow ae^1 = 0, \Rightarrow \boxed{a=0}$ because $e^1 > 0$.

10 If $a=0$ then $b\log(t) = 0$, but take any $t \neq 1$, then $\log(t) \neq 0$ and we can conclude that $\boxed{b=0}$

thus, $e^t, \log t$ are L.I.

1.2.10. $v, w \in V$, $v \neq \theta$. If v, w are L.D. show that $\exists$ a number a such that $w = av$.

Solution: By hypothesis, v, w are L.D. i.e. $c_1 v + c_2 w = 0$ and either c_1 or c_2 or both are not zero. Suppose that $c_1 = 0$ and $c_2 \neq 0$, then $c_1 v + c_2 w = \theta \Rightarrow c_2 w = \theta \Rightarrow w = \theta$. And so, we can trivially write $w = av$, where $a = 0$, i.e. $w = 0v = \theta$. Similarly, If $c_1 \neq 0$ and $c_2 = 0$, then $c_1 v + c_2 w = \theta \Rightarrow c_1 v = \theta \Rightarrow v = \theta$, But we assumed that $v \neq \theta$, so this cannot occur.

the interesting case arise when both $c_1 \neq 0$ and $c_2 \neq 0$

$c_1 v + c_2 w = 0 \Rightarrow c_1 v = -c_2 w$. Because $c_2 \neq 0$, we can divide both sides by $-c_2$ to obtain $-\frac{c_1}{c_2} v = w$

and thus we obtain $w = a v$, where $a = -\frac{c_1}{c_2}$, which is a number

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1.2.7. $3 \sin t + 5 \cos t = f(t)$ basis $\{\sin t, \cos t\}$

The coordinates are $(3,5)$. If we take a linear combination of the members of the basis: $a \sin t + b \cos t$, and we let $a = 3$ and $b = 5$, we obtain $f(t)$.

1.2.8. the function $Df(t)$ is:

$$Df(t) = (3 \sin t + 5 \cos t)' = 3 \cos t - 5 \sin t$$

And so the coordinates with respect to the basis $\{\sin t, \cos t\}$ are $(-5, 3)$

Additional Exercises:

1.- Verify that a vector subspace W of a vector space V is a vector space. (equipped with the addition and scalar multiplication of V).

Solution: A vector subspace W, as defined on p.5 of Long, satisfies 3 conditions:

(i) If $v, w \in W \Rightarrow v + w \in W$

(ii) If $v \in W$ and c a number $\Rightarrow cv \in W$

(iii) the element O of V is also in W.

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A set is a vector space if it satisfies the eigth conditions stated in p. 3 of Long. Here I will show that each condition is true for W as defined above.

(VS 1) $u, v, w \in V$: $(u+v)+w = u+(v+w)$

this is true in the subspace because the "+" operation is the same as the one in V. thus, it is true that $u', v', w' \in W$: $(u'+v')+w' = u'+(v'+w')$

Enrique Areyan - eareyan@umail.iu.edu - Homework 1 ⑥

(VS2) $\exists$ an element $\theta \in V$, such that $\forall u \in V$: $\theta + u = u + \theta = u$.
By definition of vector subspace $\theta \in W$.

(VS3) for $u \in V$, $\exists -u \in V$ such that $u + (-u) = \theta$.
to prove the existence of inverses in the subspace W, we can take an element $w \in W$ and write:

$\theta \in W$ by property (VS3) of the bigger vector space $V \Rightarrow \theta = w + (-w) \in W$

But by definition, the sum of any two elements in W is in W. therefore, assuming that $w \in W$ we can conclude that $-w \in W$.

(VS4) $\forall u, v \in V$: $u + v = v + u$. Similar to (VS1), the commutative property is inherited from V. In particular. If $w_1 + w_2 \in W$ and $w_2 + w_1 = w_1 + w_2$ then $w_2 + w_1 \in W$

(VS5) c a number, then $c(u+v) = cu + cv$.
Let d be a number and $w_1, w_2 \in W$.
TAKE $\underset{W}{\underset{\cap}{d(w_1+w_2)}} \underset{\text{from } V}{\underset{\text{inherited}}{=}} dw_1 + dw_2$ by definition (ii) $dw_1 \in W$ and $dw_2 \in W$
$\Rightarrow dw_1 + dw_2 \in W$

(VS6) If a, b are numbers then $(a+b)v = av + bv$.
this is pretty much symmetric to the preceding property:
TAKE e, d numbers and $w \in W$: $\underset{W}{\underset{\cap}{(e+d)w}} \underset{\text{inhe.}}{=} \underset{W}{\underset{\cap}{ew}} + \underset{W}{\underset{\cap}{dw}}$

(VS7) $(ab)v = a(bv)$ true using associativity of V and numbers under multiplication.

(VS8) $1 \cdot u = u$ this is trivially true inside of W as well.

Additional EXERCISE

(2) Show that if $\{v_1, \dots, v_j\}$ is a dependent subset of V then $\{v_1, \dots, v_j, v_{j+1}, \dots, v_{j+k}\}$ is also a dependent set.

Pf: the proof is by induction on K.

Base CASE: $K = 1$. We want to show that $\underset{\text{(dependent)}}{\{v_1, \dots, v_j\}} \Rightarrow \underset{\text{(dependent)}}{\{v_1, \dots, v_{j+1}\}}$

By hypothesis: $c_1 v_1 + c_2 v_2 + \dots + c_j v_j = 0 \Rightarrow$ not all c_i's are zero.

If we add $0 \cdot v_{j+1}$: $c_1 v_1 + c_2 v_2 + \dots + c_j v_j + 0 \cdot v_{j+1} = 0$ this set will still be dependent. therefore, choose $c_{j+1} = 0$ to obtain dependence for $\{v_1, v_2, \dots, v_j, v_{j+1}\}$.

Inductive step: Suppose that $\{v_1, v_2, \dots, v_j, \dots, v_{j+k}\}$ is dependent. we want to show dependence for $\{v_1, v_2, \dots, v_j, \dots, v_{j+k}, v_{j+k+1}\}$

By hypothesis: $c_1 v_1 + c_2 v_2 + \dots + c_{j+k} v_k = 0 \Rightarrow$ not all c_i's are zero

If we add $0 \cdot v_{j+k+1}$: $c_1 v_1 + c_2 v_2 + \dots + c_{j+k} v_k + 0 \cdot v_{j+k+1} = 0$.

thus, choose $c_{j+k+1} = 0$ to obtain the dependent set

$\{v_1, v_2, \dots, v_j, \dots, v_{j+k}, v_{j+k+1}\}$.

Additional EXERCISES

C3) Let $\{v_1, \dots, v_{k+1}\}$ be a basis for a Vector space V. Show that $\{v_1, \dots, v_k\}$ is a basis for the subspace of V generated by $\{v_1, \dots, v_k\}$ (Very briefly explain why theorem 3.4 is a consequence.) ?

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Pf: If $\{v_1, \dots, v_{k+1}\}$ is a basis for V, it means that this set is independent and generates all of V. Independence means that $c_1 v_1 + c_2 v_2 + \dots + c_{k+1} v_{k+1} = 0 \Rightarrow c_1 = c_2 = \dots = c_{k+1} = 0$.

So, in particular, if we set $c_{k+1} = 0$, the following will still hold: $c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0 \Rightarrow c_1 = c_2 = \dots = c_k = 0$ and $c_{k+1} = 0$

In other words, the set $\{v_1, v_2, \dots, v_k\}$ is independent. If we take all linear combinations of this set, i.e, $c_1 v_1 + c_2 v_2 + \dots + c_k v_k$, we obtain, by definition, the subspace generated by $\{v_1, v_2, \dots v_k\}$. thus, this set is L.I & generates ~~the~~ subspace, which converts it into a basis for the subspace.

SECTION 1.4.

1.- $V = \mathbb{R}^2$ $W = \langle\{(2,1)\}\rangle$; $U = \langle\{0,1\}\rangle$.

Show that $V = W \oplus U$.

Solution: By theorem 4.1, we need to check that

$U + W = V$ and $U \cap W = \{\theta\}$.

First: We want to be able to write any element of V as a sum of elements of U and W, i.e., $v = u + w$

Let $v = (x,y) \in \mathbb{R}^2$, $u = c_1(0,1) \in U$ and $w = c_2(2,1) \in W$

then $v = u + w = (2c_2, c_1 + c_2) = (x,y)$

$\Rightarrow x = 2c_2$, $y = c_1 + c_2$

$\Rightarrow \frac{x}{2} = c_2 \Rightarrow y = c_1 + \frac{x}{2} \Rightarrow c_1 = y - \frac{x}{2}$

thus, If we let $\boxed{c_2 = \frac{x}{2}}$ and $\boxed{c_1 = y - \frac{x}{2}}$ we can obtain any element v.

Second: Let z be a vector that is both on U and W:

$z = c_1(0,1)$ and $z = c_2(2,1)$

$\Rightarrow z = c_1(0,1) = c_2(2,1) \Rightarrow \begin{cases} 2c_2 = 0 \Rightarrow \boxed{c_2 = 0} \\ c_1 = c_2 \qquad\quad \Downarrow \ \boxed{c_1 = 0} \end{cases}$

$\Rightarrow \boxed{z = (0,0)}$ $\Rightarrow$ this is the only element of the intersection.

HENCE, $\mathbb{R}^2 = W \oplus U$

Now, Replace U with $U' = \langle\{(1,1)\}\rangle$

First: to show sum: let $(x,y) \in \mathbb{R}^2$, $u = c_1(1,1) \in U$, $w = c_2(2,1) \in W$.

$(x,y) = c_1(1,1) + c_2(2,1) \Rightarrow \begin{cases} x = c_1 + 2c_2 \ (2) \\ y = c_1 + c_2 \Rightarrow y - c_2 = c_1. \ (1) \end{cases}$

Replacing (1) into (2): $x = y - c_2 + 2c_2 \Rightarrow \boxed{x - y = c_2}$. Replacing this back in (1)

$y - (x-y) = c_1 \Rightarrow \boxed{c_1 = -x + 2y}$

SECond: Let z be a vector inside both U and W:

$z = c_1(1,1)$ and $z = c_2(2,1)$ $\Rightarrow z = c_1(1,1) = c_2(2,1) \Rightarrow c_1(1,1) - c_2(2,1) = 0$

$c_1 - 2c_2 = 0$ and $c_1 - c_2 = 0 \Rightarrow c_1 = c_2 \Rightarrow -c_2 = 0 \Rightarrow \boxed{c_2 = 0} \Rightarrow \boxed{c_1 = 0}$

So $U \cap W = \{\theta\}$ HENCE, $\mathbb{R}^2 = U' \oplus W$

1.4.2. $V=K^3$ for some field K, $W=\langle\{(1,0,0)\}\rangle$

$U=\langle\{(1,1,0),(0,1,1)\}\rangle$. SHOW that $V=W\oplus U$.

Pf:

First: let $v=(x,y,z)\in V=K^3$. Let $w=c_1(1,0,0)\in W$ and $u=c_2(1,1,0)+c_3(0,1,1)\in U$. then

$$(x,y,z)=c_1(1,0,0)+c_2(1,1,0)+c_3(0,1,1)$$

$$\Rightarrow \begin{cases} x=c_1+c_2 & \Rightarrow x=c_1+y-z \Rightarrow \boxed{c_1=x-y+z} \\ y=c_2+c_3 & \Rightarrow y=c_2+z \Rightarrow \boxed{c_2=y-z} \\ \boxed{z=c_3} \end{cases}$$

thus, we can write any vector in V with the choices of c_1, c_2 and c_3 as above.

Second: let t be a vector inside U and W. then:

$$\begin{array}{l} t=c_1(1,0,0) \\ t=c_2(1,1,0)+c_3(0,1,1) \end{array} \Rightarrow \left\{ \begin{array}{l} c_1(1,0,0)=c_2(1,1,0)+c_3(0,1,1) \\ \Downarrow \\ c_1(1,0,0)-c_2(1,1,0)-c_3(0,1,1)=\theta \end{array}\right.$$

$$\Rightarrow \begin{cases} c_1-c_2=0 \\ -c_2-c_3=0 \\ -c_3=0 \end{cases} \Rightarrow c_3=0 \Rightarrow c_2=0 \Rightarrow c_1=0$$

thus, the only vector in the intersection is θ.

Because First & Second we conclude that $V=W\oplus U$.

1.4.3 $A,B\in\mathbb{R}^2$ and $A\neq 0, B\neq 0$. If there is no number c such that $cA=B$, show that A,B form a basis of $\mathbb{R}^2$, and that $\mathbb{R}^2$ is a direct sum of the subspaces generated by A and B.

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(i) to show that $\{A,B\}$ form a basis, we need to show that these are L.I and generate V.

(1.i) Linear Independence. we want to show $c_1A+c_2B=0 \Rightarrow c_1=c_2=0$. Suppose to the contrary that $\{A,B\}$ are dependent. then, $c_1A+c_2B=0 \Rightarrow$ not all c_i's $=0$. Suppose that $c_1\neq 0$ and $c_2=0$ then $c_1A=0 \Rightarrow A=0$ but we assumed $A\neq 0$, so this is not possible. On the other hand, suppose that $c_1=0$ and $c_2\neq 0$. then $c_2B=0 \Rightarrow B=0$ but we assumed $B\neq 0$, so this is not possible. Finally, suppose that both $c_1\neq 0$ and $c_2\neq 0$.

then, $c_1A+c_2B=0 \Rightarrow c_1A=-c_2B$. We can divide by $-c_2$:

$-\frac{c_1}{c_2}A=B$. But, by hypothesis we assumed that there is no number c such that $cA=B$, in this case we found $c=-\frac{c_1}{c_2}$, which is not possible.

therefore, $\{A,B\}$ is an independent set.

(2.i) we know that $\mathbb{R}^2$ is of dimension 2. Using theorem 3.4. and the fact we just proved in (1.i), we can conclude that $\{A,B\}$ constitute a basis of V.

(ii) Direct sum: need to show (1-ii) $\mathbb{R}^2=\langle\{A\}\rangle+\langle\{B\}\rangle$ and (2-ii) $\langle\{A\}\rangle\cap\langle\{B\}\rangle=\theta$.

(1-ii) From (2.i) we know that $\{A,B\}$ is a basis of $\mathbb{R}^2$, so any vector in it can be expressed as the sum $v=c_1A+c_2B$, where $c_1,c_2\in\mathbb{R}^2$, and $c_1A\in\langle\{A\}\rangle$ $c_2B\in\langle\{B\}\rangle$.

morover, because $\{A,B\}$ is a basis, then v's representation is unique (theorem 2.1) and the direct sum holds

$\Rightarrow \boxed{\mathbb{R}^2=\langle\{A\}\rangle\oplus\langle\{B\}\rangle}$.

1.4.4. If $\{u_1,\dots,u_r\}$ is a basis of U and $\{w_1,\dots,w_s\}$ is a basis of W. what is a basis of $U\times W$?

Pf:

By definition, $U\times W=\{(u,w)\mid u\in U \text{ and } w\in W\}$

Let $v\in U\times W$, then $v=(u,w)$, where $u\in U, w\in W$

If $\{u_1,\dots,u_r\}$ is a basis of U, then we can write any element $u\in U$, uniquely as $u=x_1u_1+x_2u_r+\dots+x_ru_r$. Similarly, If $\{w_1,\dots,w_s\}$ is a basis of W, then we can write any element $w\in W$ uniquely as $w=y_1w_1+y_2w_2+\dots+y_sw_s$ (Here, x_i's and y_i's are numbers)

therefore, we can write any element $(u,v) \in U \times W$, uniquely
as $(u,v) = (x_1 u_1 + \dots + x_r u_r, y_1 w_1 + \dots + y_s w_s)$, and $\{u_1, \dots, u_r, w_1, \dots, w_s\}$
form a basis of $U \times V$. thus, $\text{Dim } U \times W = r + s = \dim(U) + \dim(W)$.
we can think of the basis for $U \times W$ as the vector form by
$$B = \{(u_1, \theta), (u_2, \theta), \dots, (u_r, \theta), (\theta, w_1), (\theta, w_2), \dots, (\theta, w_s)\}$$

Any element in $U \times W$ can be writen as linear combinations of B:

$$(u,w) = \left(\sum_{i=1}^{r} x_i u_i, \theta\right) + \left(\theta, \sum_{i=1}^{s} y_i w_i\right) \quad B \text{ generates } U \times W$$

the elements in $U \times W$ are independent:

$$(\theta, \theta) = \left(\sum_{i=1}^{r} x_i u_i, \theta\right) + \left(\theta, \sum_{i=1}^{s} y_i w_i\right)$$

$$\Rightarrow \quad \theta = \sum_{i=1}^{r} x_i u_i \Rightarrow x_i = 0 \text{ for } i = 1, \dots, r \quad \text{By hypothesis}$$

$$\theta = \sum_{i=1}^{s} y_i w_i \Rightarrow y_i = 0 \text{ for } i = 1, \dots, s \quad \text{By hypothesis}$$

B is liner independent and generates $U \times W$ ~~so~~ is a basis.

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