Enrique Areyan - M409 - Homework 3

Section 5.1:

1) Let V be a vector space with scalar product. Show that $\langle \theta, v\rangle = 0$ $\forall v \in V$.

Pf: Let v be an arbitrary element in V. then:

$$\langle \theta, v\rangle = \langle \theta+\theta, v\rangle \quad \text{By definition of } \theta \in V$$
$$= \langle \theta, v\rangle + \langle \theta, v\rangle \quad \text{Property SP2.}$$

$\Rightarrow \langle \theta, v\rangle = \langle \theta, v\rangle + \langle \theta, v\rangle$. Substract the scalar $\langle \theta, v\rangle$ from both sides:

$$\langle \theta, v\rangle - \langle \theta, v\rangle = \langle \theta, v\rangle + (\langle \theta, v\rangle - \langle \theta, v\rangle) \quad \text{Associativity of the field associated with } V.$$
$$0 = \langle \theta, v\rangle + 0$$

Hence, $\langle \theta, v\rangle = 0$.

An alternative proof is

$$\langle \theta, v\rangle = \langle v - v, v\rangle \quad \text{By definition of inverses in } V$$
$$= \langle v, v\rangle - \langle v, v\rangle \quad \text{Property SP2}$$
$$= 0$$

2) Assume that the scalar product is positive definite. Let $v_1, \ldots, v_n$ be non-zero elements which are mutually perpendicular, that is $\langle v_i, v_j\rangle = 0$ if $i \neq j$. Show that they are linearly independent.

Pf: By hypothesis, $\langle v, v\rangle \geq 0$ $\forall v \in V$ and $\langle v, v\rangle = 0$ if $v \neq \theta$

Proof by contradiction: Suppose that $v_1, \ldots, v_n$ are no linearly independent. then, without loss of generality, we can write one element say v_k, as a linear combination of the others, i.e.,

$$v_k = \sum_{i \neq k} d_i v_i$$

By definition, $\langle v_k, v_k\rangle > 0$ because we assume all $v_i \neq 0$ and $\langle , \rangle$ pos. definite

But $\langle v_k, v_k\rangle = \langle v_k, d_1 v_1 + \cdots + d_{k-1} v_{k-1} + d_{k+1} v_{k+1} + \cdots + d_n v_n\rangle$

$= \langle v_k, d_1 v_1\rangle + \langle v_k, d_2 v_2\rangle + \cdots + \langle v_k, d_n v_n\rangle$ Property SP2

$= d_1 \langle v_k, v_1\rangle + d_2 \langle v_k, v_2\rangle + \cdots + d_n \langle v_k, v_n\rangle$ Property SP3

$= d_1 \cdot 0 + d_2 \cdot 0 + \cdots + d_n \cdot 0$ By hypothesis $\langle v_i, v_j\rangle = 0$ if $i \neq j$

$= 0 + 0 + \cdots + 0$

$= 0$

$\Rightarrow \langle v_k, v_k\rangle = 0$ which contradicts the fact that $\langle v_k, v_k\rangle > 0$

Hence, $v_1, \ldots, v_n$ is linearly independent.

(3) SP1: $\forall v,w \in V: \langle v,w\rangle = \langle w,v\rangle$.

Let $V = \mathbb{K}^n$. Let $X, Y \in \mathbb{K}^n$ (colum vectors) then

$\langle X, Y\rangle = {}^tX M Y$ By definition of $\langle X,Y\rangle$

Note that ${}^tXMY$ is a 1×1 matrix, i.e, a scalar. thus,

$({}^tXMy)^t = {}^tXMy$ (the transpose of a number is the number itself)

We can group terms like ${}^t({}^tXMy) = {}^t(x^t(My))$

By theorem 3.3, we obtain ${}^t({}^tX(My)) = {}^t(My)^t({}^tX)$.

Taking the transpose twice has no effect, thus $({}^tX)^t = X$

Also, we can apply theorem 3.3 on the term ${}^t(My) = y^t M^t$

Hence, ${}^t(My)^t({}^tX) = {}^ty^tMX$. But, By hypothesis ${}^tM = M$

thus, ${}^ty^tMx = {}^tyMx$

therefore, $\langle x,y\rangle = {}^txMy = {}^tyMx = \langle y,x\rangle$ Property SP1 holds.

SP2: $u,v,w \in V \Rightarrow \langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$

Let $X,Y,Z \in \mathbb{K}^n$. then

$\langle X, Y+Z\rangle = {}^tXM(Y+Z)$ By definition of $\langle\,,\rangle$

$= {}^tX(M(Y+Z))$ Associativity

$= {}^tX(MY+MZ)$ Matrix distributivity

$= {}^tXMY + {}^tXMZ$ " " " "

$= \langle X,Y\rangle + \langle X,Z\rangle$ By definition of $\langle\,,\rangle$

Hence, property SP2 holds.

SP3: If $x \in \mathbb{K}$, then, $\langle xu,v\rangle = x\langle u,v\rangle$ and $\langle u,xv\rangle = x\langle u,v\rangle$

Let $c \in \mathbb{K}$ and $X, y \in \mathbb{K}^n$. then:

$\langle cX,Y\rangle = {}^t(cx)My =$ By theorem 3.3 $= {}^tx\,{}^tc\,My$. But the transpose of a number is the number itself thus $= {}^tx\,c\,My =$

By theorem 3.1 $= c\,{}^txMy = c\langle X,Y\rangle$ By definition of $\langle\,,\rangle$

Likewise, $\langle X, cY\rangle = {}^tXMcy =$ (theorem 3.1) $= c\,{}^tXMy = c\langle x,y\rangle$.

Hence, SP3 Holds.

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Give an example of a 2x2 matrix M such that the product is not positive definite

Solution: Let $M \in \mathbb{R}^{2\times 2}$ be $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

then $\langle x, x \rangle = (0\ 1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0\ 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 + 0 = 0$.

But $x \neq \theta$ Hence $\langle , \rangle$ with $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not pos. def.

Additional Exercises:

1.- Let $a, b, c \in \mathbb{R}$. Define $B: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$B((v_1, v_2), (w_1, w_2)) = a v_1 w_1 + b v_1 w_2 + c v_2 w_1 + d v_2 w_2$$

(i) Show that B is a bilinear form.

Pf: Let $x, y, z \in \mathbb{R}^2$ and $s, t \in \mathbb{R}$. Then: we want to prove that: $B((sx+ty), z) \stackrel{?}{=} s B(x,z) + t B(y,z)$

TAKE: $B((sx+ty), z) = B((s(x_1, x_2) + t(y_1, y_2)), (z_1, z_2))$ .. Def x, y, z

$= B((sx_1 + ty_1, sx_2 + ty_2), (z_1, z_2))$ .... Def '+' in $\mathbb{R}^2$ and scalar multiplication

Def B.. $= a(sx_1+ty_1)(z_1) + b(sx_1+ty_1)(z_2) + c(sx_2+ty_2)(z_1) + d(sx_2+ty_2)(z_2)$

associativity $= asx_1z_1 + aty_1z_1 + bsx_1z_2 + bty_1z_2 + csx_2z_1 + cty_2z_1 + dsx_2z_2 + dty_2z_2$

commutativity of $\mathbb{R}$ $= s(ax_1z_1 + bx_1z_2 + cx_2z_1 + dx_2z_2) + t(ay_1z_1 + by_1z_2 + cy_2z_1 + dy_2z_2)$

Def B.. $= s B(x, z) + t B(y, z)$.

Hence, $B((sx+ty), z) = sB(x,z) + tB(y,z)$, B is bilinear.

(ii) Determine which choices of a, b, c, d make B

(a) positive definite. By definition, B is pos def iff. $\forall x \in \mathbb{R}^2$ $B(x,x) \geq 0$ and $B(x,x) = 0$ iff $x = \theta$.

We have two cases:

(i) Suppose $x \in \mathbb{R}^2$ is such that $x = \theta$. then,

$B(x, x) = B((0,0), (0,0)) = 0$ for any choice of a, b, c, d.

(ii) Suppose $x \in \mathbb{R}^2$ is such that $x \neq \theta$. then,

$B(x, x) = B((x_1, x_2), (x_1, x_2)) = ax_1^2 + (b+c)(x_1 x_2) + d x_2^2$

By definition of B. In order to be positive definite,

$ax_1^2 + dx_2^2 + (b+c)(x_1x_2) > 0$ for any choice of x_1 and x_2.

Particularly,

If $x_1 = 1$ and $x_2 = 0$, then
$B((1,0),(1,0)) = a > 0$, thus $a > 0$.

If $x_1 = 0$ and $x_2 = 1$, then
$B((0,1),(0,1)) = d > 0$ thus $d > 0$

If $x_1 = -1$ and $x_2 = 1$, then
$B((-1,1),(-1,1)) = a + d - (b+c) > 0$ But from previous choices of x_1 and x_2, we know that both a and d are greater than zero

Hence, $a + d > b + c$.

thus, in order for B to be pos. definite, the following conditions must hold. $a > 0$ and $d > 0$ and $a + d > b + c$.

(b) nondegenerate. By definition, B is nondegenerate iff given $x \in \mathbb{R}^2$ if $B(x,y) = 0$ $\forall y$ then $x = 0$

Let $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$. then
$B(x,y) = B((x_1,x_2),(y_1,y_2)) = a x_1 y_1 + b x_1 y_2 + c x_2 y_1 + d x_2 y_2$.

Now, if $B(x,y) = 0 \Rightarrow a x_1 y_1 + b x_1 y_2 + c x_2 y_1 + d x_2 y_2 = 0$ $(*)$

In order for B to be nondegenerate, $(*)$ has to hold for any (y_1, y_2) and imply that $x_1 = x_2 = 0$.

Pick $(y_1,y_2) = (1,0)$ then, $(*)$ becomes:
$a x_1 + c x_2 = 0$ (Eq1)
Pick $(y_1,y_2) = (0,1)$ then, $(*)$ becomes
$b x_1 + d x_2 = 0$ (Eq2)

Hence, $a x_1 + c x_2 = b x_1 + d x_2 \Rightarrow (a-b) x_1 + (c-d) x_2 = 0$

therefore, If $a \neq b$ and $c \neq d \Rightarrow x_1 = x_2 = 0$, which shows that in order for B to be non degenerate, $a \neq b$ and $c \neq d$ at the same time.

(c) Anti-symmetric. By definition, B is antisymmetric if given $x, y \in \mathbb{R}^2$ then $B(x,y) = -B(x,y)$. $\forall x, y \in \mathbb{R}^2$. $\Rightarrow$
$a x_1 y_1 + b x_1 y_2 + c x_2 y_1 + d x_2 y_2 = -a x_1 y_1 - b x_1 y_2 - c x_2 y_1 - d x_2 y_2$
$2a x_1 y_1 + 2b x_1 y_2 + 2c x_2 y_1 + 2d x_2 y_2 = 0$

Let $x_1 = 1, x_2 = 0, y_1 = 1, y_2 = 0$

$2a = 0 \Rightarrow a = 0$

Let $x_1 = 1, x_2 = 0, y_1 = 0, y_2 = 1$

$2b = 0 \Rightarrow b = 0$

Let $x_1 = 0, x_2 = 1, y_1 = 1, y_2 = 0$

$2c = 0 \Rightarrow c = 0$

Let $x_1 = 0, x_2 = 1, y_1 = 0, y_2 = 1$

$2d = 0 \Rightarrow d = 0$

Hence, the only way B can be antisymmetric is if $a = b = c = d = 0$.
But this is the trivial bilinear form $B(x,y) = 0$.

(d) Symmetric. By definition, B is symmetric iff $\forall x, y \in \mathbb{R}^2$
$B(x,y) = B(y,x) \Rightarrow ax_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2 =$
$ay_1x_1 + by_2x_1 + cy_1x_2 + dy_2x_2$

But, using commutative of real numbers, we can write the right-hand side of the above equation as the left-hand size. Thus, B is symmetric regardless of any choice for $a, b, c,$ or d.

Additional exercises

2) Let $\langle , \rangle$ be a positive definite scalar product on a real vector space V and let $\|\cdot\|$ be the associated norm. Prove that for all v and w belonging to V, we have that
$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

Pf. $\|v+w\|^2 + \|v-w\|^2 = \langle v+w, v+w\rangle + \langle v-w, v-w\rangle$ Definition of norm

linearity of $\langle , \rangle$ $= \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle + \langle v,v\rangle + \langle v,-w\rangle + \langle -w,v\rangle + \langle -w,-w\rangle$

grouping terms ? $= 2\langle v,v\rangle + 2\langle v,w\rangle + 2\langle w,w\rangle + 2\langle v,-w\rangle$ ... Linearity of $\langle , \rangle$

symmetry of $\langle , \rangle$ $= 2\langle v,v\rangle + 2\langle v,w\rangle - 2\langle v,w\rangle + 2\langle w,w\rangle$

$= 2\langle v,v\rangle + 2\langle w,w\rangle$ .... Definition of norm

$= 2\|v\|^2 + 2\|w\|^2$

Hence, $\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$ //

Section 5.2

(3) V = vector space of continuous real-valued functions on the interval $[0,1]$. We define the scalar product of two such functions f, g by the rule $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. Using standard properties of the integral, verify that this is a scalar product.

<u>Solution</u>: A scalar product is a symmetric bilinear form. Check these two properties:

(i) Symmetry Let $f, g \in V$ then:

$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$ By definition of $\langle , \rangle$ on V.

$= \int_0^1 g(t)f(t)\,dt$ By commutativity of real numbers.

$= \langle g, f\rangle$ By definition of $\langle , \rangle$ on V.

Hence, $\langle f, g\rangle = \langle g, f\rangle$ is symmetric.

(ii) Bilinearity. Let $s, s' \in \mathbb{R}$ and $f, g, h \in V$. then:

$\langle sf + s'g, h\rangle = \int_0^1 (sf(t) + s'g(t))h(t)\,dt$ By definition of $\langle , \rangle$ on V

$= \int_0^1 sf(t)h(t) + s'g(t)h(t)\,dt$ distributivity

$= \int_0^1 sf(t)h(t)\,dt + \int_0^1 s'g(t)h(t)\,dt$ standard property of the integral

$= s\int_0^1 f(t)h(t)\,dt + s'\int_0^1 g(t)h(t)\,dt$ " " " " "

$= s\langle f, h\rangle + s'\langle g, h\rangle$ By definition of $\langle , \rangle$ on V

Hence, $\langle sf + s'g, h\rangle = s\langle f, h\rangle + s'\langle g, h\rangle$

Likewise (using the same reasoning as before:

$\langle f, sg + s'h\rangle = \int_0^1 f(t)(sg(t) + s'h(t))\,dt = \int_0^1 f(t)sg(t) + f(t)s'h(t)\,dt$

$= \int_0^1 sf(t)g(t)\,dt + \int_0^1 s'f(t)h(t)\,dt = s\int_0^1 f(t)g(t)\,dt + s'\int_0^1 f(t)h(t)\,dt$

$= s\langle f, g\rangle + s'\langle f, h\rangle$. Hence, $\langle f, sg + s'h\rangle = s\langle f, g\rangle + s'\langle f, h\rangle$

Therefore, $\langle , \rangle$ is a bilinear form. It is also symmetric, so it is a scalar product.

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Additional Exercises.

(1). Let V and $\langle , \rangle$ be defined as in exercise 3 in section 5.2

Let $f \in V$ be defined by $f(x) = 1 \ \forall x \in [0,1]$.

(i) Describe the subspace orthogonal to $\{f\}$ in terms of integration

Solution: By definition $\{f\}^\perp = \{g \in V \mid \langle f, g \rangle = 0\}$

But $\langle f, g \rangle = 0 \iff \int_0^1 f(t) g(t)\, dt = 0$ By def of $\langle , \rangle$

$= \int_0^1 1 \cdot g(t)\, dt = \int_0^1 g(t)\, dt = 0$ By def of f.

Hence, $\{f\}^\perp = \{g \in V \mid \int_0^1 g(t)\, dt = 0\}$

(ii) Describe the orthogonal projection onto $\langle \{f\} \rangle$ in terms of integration

Solution: By definition $P_{\langle \{f\} \rangle} : V \to V$ is $P_{\langle \{f\} \rangle}(g) = \frac{\langle g, f \rangle}{\langle f, f \rangle} \cdot f$

$= \frac{\int_0^1 g(t) f(t)\, dt}{\int_0^1 f(t) f(t)\, dt} f$. But, $\int_0^1 f(t) f(t)\, dt = \int_0^1 1 \cdot 1\, dt = t \Big|_0^1 = 1 - 0 = 1$

Fundamental theorem of calculus.

AND $\int_0^1 g(t) f(t)\, dt = \int_0^1 g(t) 1 \cdot dt = \int_0^1 g(t)\, dt$

Hence, $P_{\langle \{f\} \rangle}(g) = \left(\int_0^1 g(t)\, dt \right) \cdot f(x) = \boxed{\int_0^1 g(t)\, dt}$

(iii) what is the kernel of this projection?

Solution. By definition, $Ker(P_{\langle \{f\} \rangle}) = \{g \in V \mid P_{\langle \{f\} \rangle}(g) = 0\}$

By previous work (i) & (ii) $Ker(P_{\langle \{f\} \rangle}) = \{g \in V \mid \int_0^1 g(t)\, dt = 0\}$

(2) Let V and $\langle , \rangle$ be defined as in exercise 3 in section 5.2.

Let $A \subset V$ be $A = \{\sin(k\pi x) \mid k \text{ is an integer}\}$. Show that this set is an orthogonal set.

Pf: By definition, $A \subset V$ is orthogonal w.r.t $\langle , \rangle$ iff $\forall f, g \in A$, with $f \neq g$, $\langle f, g \rangle = 0$.

Let $f \in A$ and $g \in A$. then:

$$\langle f, g \rangle = \int_0^1 f(t) g(t) \, dt$$ By definition of $\langle , \rangle$

$$= \int_0^1 \sin(k\pi t) \sin(k'\pi t) \, dt$$ Using trigonometric identity and linearity of integral

$$= \frac{1}{2} \int_0^1 \cos(k\pi t - k'\pi t) - \cos(k\pi t + k'\pi t) \, dt$$ Reflecting in $\theta = \pi$

$$= \frac{1}{2} \int_0^1 \cos(\overbrace{k\pi t - k'\pi t}^{\alpha}) + \cos(\overbrace{\pi - k\pi t - k'\pi t}^{\beta}) \, dt$$

α and β are opposite angle, thus $\cos(\alpha) = -\cos(\beta)$

Hence $\cos(\alpha) + \cos(\beta) = 0$

$$= \frac{1}{2} \int_0^1 0 \, dt = 0$$

Hence, $\langle f, g \rangle = 0$ if $f, g \in V$ and $f \neq g$.

$\Rightarrow$ A is an orthogonal set.

(3) Show that the integral of $\sin(x)/x$ over the interval $[1,2]$ is no greater that the product of the square root of the integral of $\sin(x)^2$ over $[1,2]$ and the square root of the integral of $(1/x)^2$ over $[1,2]$ (Hint: use the Cauchy-Schwartz inequality)

$$\int_1^2 \frac{\sin(x)}{x} \, dx \leq \sqrt{\int_1^2 \sin(x)^2} \cdot \sqrt{\int_1^2 \frac{1}{x^2} \, dx}$$

Pf: To prove this, first define V = vector space of continuos real-valued functions on the interval $[1,2]$. Define the scalar product on V as $\langle f, g \rangle = \int_1^2 f(t) g(t) \, dt$. In exercise 3 for section 5.2, it was shown that a very similar space and $\langle , \rangle$ is a scalar product define over $[1,2]$ Using the same proof, it is trivial that for functions the product $\langle , \rangle$ is a scalar product. (The proof is the same, just replace $[0,1]$ with $[1,2]$.) Furthermore, $\langle , \rangle$ is positive definite Because: let $f \in V$: then $\langle v, v \rangle = \int_1^2 f(t) f(t) \, dt = \int_1^2 f(t)^2 \, dt$

But $f(t)^2 \geq 0$ and hence, the integral is alway greater than zero. (the are under the curve of $f(t)^2$ is always above the t-axis).

Furthermore, if $\langle v, v \rangle = 0 \Leftrightarrow \int_1^2 f(t)^2 \, dt \Leftrightarrow f(t)^2 = 0 \Leftrightarrow f(t) = 0$.

Now, given that $\langle , \rangle$ is a positive definite scalar product, we can apply Cauchy-Schwartz.

Let $f, g \in V$ be $f(x) = \sin(x)$ and $g(x) = \frac{1}{x}$

then, by cauchy-schwartz:

$$|\langle f, g\rangle| \leq \|f\| \cdot \|g\| \quad \ldots \quad \text{THEOREM}$$

$$\left|\langle \sin(x), \tfrac{1}{x}\rangle\right| \leq \|\sin(x)\| \cdot \|\tfrac{1}{x}\| \quad \ldots \quad \text{Definition of } f, g$$

$$\left|\int_1^2 \sin(x) \cdot \frac{1}{x}\, dx\right| \leq \sqrt{\langle \sin(x), \sin(x)\rangle} \cdot \sqrt{\langle \tfrac{1}{x}, \tfrac{1}{x}\rangle} \quad \ldots \quad \text{Definitions of } \langle , \rangle \text{ and } \|\cdot\|$$

$$\left|\int_1^2 \frac{\sin(x)}{x}\, dx\right| \leq \sqrt{\int_1^2 \sin(x)^2\, dx} \cdot \sqrt{\int_1^2 \frac{1}{x^2}\, dx} \quad \ldots \quad \text{Definition of } \langle , \rangle$$

Hence, $$\int_1^2 \frac{\sin(x)}{x}\, dx \leq \sqrt{\int_1^2 \sin(x)^2\, dx} \cdot \sqrt{\int_1^2 \frac{1}{x^2}\, dx} \quad \ldots \quad \text{Definition of abs. value } (|x| < c \Rightarrow x < c \text{ and } -x < c)$$

which is what we wanted to prove.

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(4) Let $B: V \times V \to K$ be a symmetric bilinear form and let u and w be vectors in V such that $B(u,u)$ and $B(w,w)$ are not equal to zero. Let $P: V \to V$ denote the orthogonal projection on the line spanned by u and let $Q: V \to V$ denote the orthogonal projection onto the line spanned by w. Prove that if u is orthogonal to w, then the composition $P \circ Q$ is the zero mapping, i.e. $P(Q(v)) = 0$ for all $v \in V$.

Pf: $v \in V$: $P(Q(v)) = P\left(\frac{\langle v,w\rangle}{\langle w,w\rangle} w\right) \quad \ldots$ By definition of Q

$= \frac{\langle v,w\rangle}{\langle w,w\rangle} P(w) \quad \ldots$ linearity of Q

$= \frac{\langle v,w\rangle}{\langle w,w\rangle}\left(\frac{\langle w,u\rangle}{\langle u,u\rangle} \cdot u\right) = (*) \quad \ldots$ By definition of P

Note that this expression is well-defined because $\langle w,w\rangle \neq 0$ and $\langle u,u\rangle \neq 0$. Also, [illegible] hypothesis $\langle u,w\rangle = \langle w,u\rangle = 0$. thus,

$\circledast = \frac{\langle v, w\rangle \cdot 0}{\langle w,w\rangle\langle u,u\rangle} u = 0 \cdot u = 0$, regardless of what v is.

Hence, this proves that $P(Q(v)) = 0 \quad \forall v \in V$

Section 5.2

(1) Find an orthonormal basis for the subspace of $\mathbb{R}^3$ generated by the following vectors:

(a) $\{(1,1,-1), (1,0,1)\}$.

Solution: Note that these vectors are orthogonal, i.e.,

$$\langle (1,1,-1), (1,0,1)\rangle = 1.1 + 1\,0 + (-1).1 = 1 - 1 = 0$$

thus, we need only to scale them to unit vectors.

Let $v_1 = (1,1,-1)$ and $v_2 = (1,0,1)$ then

$$\hat{u}_1 = \frac{v_1}{\|v_1\|} = \frac{\langle 1,1,-1\rangle}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\langle 1,1,-1\rangle$$ and

$$\hat{u}_2 = \frac{v_2}{\|v_2\|} = \frac{\langle 1,0,1\rangle}{\sqrt{1^2+0^2+1^2}} = \frac{1}{\sqrt{2}}\langle 1,0,1\rangle$$

the orthonormal basis is $\left\{\frac{1}{\sqrt{3}}(1,1,-1), \frac{1}{\sqrt{2}}(1,0,1)\right\}$

(b) $(2,1,1)$ and $(1,3,-1)$

These are not orthogonal vectors, i.e. $\langle (2,1,1),(1,3,-1)\rangle = 2+3-1 = 4 \neq 0$.

Hence, we apply Gram-Schmidt: Let $v_1 = (2,1,1)$ and $v_2 = (1,3,-1)$

$$u_1 = v_1. \qquad u_2 = v_2 - P_{\langle v_1\rangle}(v_2) = (1,3,-1) - \frac{\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = (1,3,-1) - \frac{4}{6}(2,1,1)$$

$$u_2 = \tfrac{1}{3}(-1,7,-5).$$ $\{u_1, u_2\}$ form an orthogonal basis.

To obtain an orthonormal basis, we scale each vector:

$$\hat{u}_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{6}}(2,1,1). \qquad \hat{u}_2 = \frac{u_2}{\|u_2\|} = \frac{\frac{1}{3}(-1,7,-5)}{\sqrt{\frac{1}{9}+\frac{49}{9}+\frac{25}{9}}} = \frac{\frac{1}{3}}{\sqrt{\frac{75}{9}}}(-1,7,-5) = \frac{1}{\sqrt{75}}(-1,2,-5)$$

the orthonormal basis is $\{\hat{u}_1, \hat{u}_2\} = \left\{\frac{1}{\sqrt{6}}(2,1,1), \frac{1}{\sqrt{75}}(-1,7,-5)\right\}$.

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Section 5.2.

(4) Let V be the subspace of functions generated by the two functions f, g such that $f(t)=t$ and $g(t)=t^2$. Find an orthonormal basis for V.

solution: First note that $\{t, t^2\}$ is not an orthogonal set w.r.t the scalar product given in 3. ⇒ by Fundamental thm. of Calculus

$$\langle t, t^2\rangle = \int_0^1 t\cdot t^2\,dt = \int_0^1 t^3\,dt = \frac{t^4}{4}\Big|_0^1 = \frac{1}{4} - \frac{0}{4} = \frac{1}{4}$$

Hence, we can apply gram-schmidt. Let $v_1 = t$ and $v_2 = t^2$

$$U_1 = V_1 = t \qquad U_2 = V_2 - P_{\langle v_1\rangle} v_2 = t^2 - \frac{\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = t^2 - \frac{\int_0^1 t^2\, t\,dt}{\int_0^1 t\cdot t\,dt}\, t = t^2 - \frac{\int_0^1 t^3 dt}{\int_0^1 t^2 dt}\, t$$

We already computed $\int_0^1 t^3 dt = \frac{1}{4}$

Also, $\int_0^1 t^2 dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}$. thus, $U_2 = t^2 - \frac{\frac{1}{4}}{\frac{1}{3}} t = t^2 - \frac{3}{4} t$.

we can check that indeed U_1 is orthogonal to U_2

$$\langle U_1, U_2\rangle = \langle t, t^2 - \tfrac{3}{4}t\rangle = \int_0^1 t(t^2 - \tfrac{3}{4}t)\,dt = \int_0^1 t^3 - \tfrac{3}{4}t^2\,dt$$ By linearity of the integral:

$$= \int_0^1 t^3 dt - \frac{3}{4}\int_0^1 t^2 dt = \frac{1}{4} - \frac{3}{4}\left(\frac{1}{3}\right) = \frac{1}{4} - \frac{1}{4} = 0.$$

To obtain an orthonormal basis we need to scale each vector:

$$\hat{U}_1 = \frac{U_1}{\|U_1\|} = \frac{t}{\sqrt{\langle t, t\rangle}} = \frac{t}{\sqrt{\int t^2 dt}} = \frac{t}{\sqrt{\frac{1}{3}}} = \frac{t}{\frac{1}{\sqrt{3}}} = \sqrt{3}\, t$$

$$\hat{U}_2 = \frac{U_2}{\|U_2\|} = \frac{t^2 - \frac{3}{4}t}{\sqrt{\langle U_2, U_2\rangle}} = \qquad \|U_2\| = \sqrt{\langle U_2, U_2\rangle} = \sqrt{\int_0^1 (t^2 - \tfrac{3}{4}t)^2 dt} = \sqrt{\int_0^1 t^4 - \tfrac{3}{2}t^3 + \tfrac{9}{16}t^2\,dt}$$

$$= \sqrt{\int_0^1 t^4 dt - \frac{3}{2}\int_0^1 t^3 dt + \frac{9}{16}\int_0^1 t^2 dt} = \sqrt{\frac{1}{5} - \frac{3}{2}\left(\frac{1}{4}\right) + \frac{9}{16}\left(\frac{1}{3}\right)} = \sqrt{\frac{1}{5} - \frac{3}{8} + \frac{3}{16}} = \sqrt{\frac{16 - 30 + 15}{16\cdot 5}} = \sqrt{\frac{1}{16\cdot 5}} = \frac{1}{4\sqrt{5}}$$

$\Rightarrow \hat{U}_2 = 4\sqrt{5}(t^2 - \frac{3}{4}t)$. the orthonormal basis is $\{\sqrt{3}\,t, 4\sqrt{5}(t^2 - \frac{3}{4}t)\}$

(5) Let V be the subspace generated by the three functions $1, t, t^2$. Find an orthonormal basis for V.

solution: Let $V_1 = 1$, $V_2 = t$, $V_3 = t^2$. Apply gram-schmidt:

$$U_1 = V_1. \quad U_2 = V_2 - P_{\langle U_1\rangle} V_2 = t - \frac{\langle V_2, V_1\rangle}{\langle V_1, V_1\rangle} V_1 = t - \frac{\int_0^1 t\,dt}{\int_0^1 1\,dt}\cdot 1 = \boxed{t - \frac{1}{2} = U_2}$$

$$U_3 = V_3 - P_{\langle U_2\rangle} V_3 - P_{\langle U_1\rangle} V_3 = t^2 - \frac{\langle V_3,U_2\rangle}{\langle U_2,U_2\rangle} U_2 - \frac{\langle V_3,U_1\rangle}{\langle U_1,U_1\rangle} U_1$$

Note that U_1 is already a unit vector:

$$\langle U_1,U_1\rangle = \langle 1,1\rangle = \int_0^1 1\cdot 1\,dt = t\Big|_0^1 = 1-0 = 1$$

Also, $\langle V_3,U_1\rangle = \langle t^2,1\rangle = \int_0^1 t^2\,dt = \frac13$

$$\langle U_2,U_2\rangle = \langle t-\tfrac12, t-\tfrac12\rangle = \int_0^1 (t-\tfrac12)^2\,dt = \int_0^1 t^2 - t + \tfrac14\,dt$$

$$= \int_0^1 t^2\,dt - \int_0^1 t\,dt + \frac14\int_0^1 1\,dt = \frac13 - \frac12 + \frac14 = \frac{4-6+3}{12} = \frac1{12}$$

$$\langle V_3,U_2\rangle = \langle t^2, t-\tfrac12\rangle = \int_0^1 t^2(t-\tfrac12)\,dt = \int_0^1 t^3 - \tfrac12 t^2\,dt$$

$$= \int_0^1 t^3\,dt - \frac12\int_0^1 t^2\,dt = \frac{t^4}{4}\Big|_0^1 - \frac12\left[\frac{t^3}{3}\right]_0^1 = \frac14 - \frac16 = \frac{6-4}{24} = \frac{2}{24} = \frac1{12}$$

Hence, $U_3 = t^2 - \dfrac{\frac1{12}}{\frac1{12}}(t-\tfrac12) - \dfrac{\frac13}{1}\,1 = t^2 - t + \frac12 - \frac13 = \boxed{t^2 - t + \frac16 = U_3}$

the set $\{1, t-\frac12, t^2-t+\frac16\}$ is an orthogonal basis.

to obtain an orthonormal set, we need to normalize U_2 and U_3. (U_1 is already a normal vector)

$$\hat U_2 = \frac{U_2}{\|U_2\|} = \frac{t-\frac12}{\sqrt{\langle U_2,U_2\rangle}} = \frac{t-\frac12}{\sqrt{\frac1{12}}} = \frac{t-\frac12}{\frac{1}{2\sqrt3}} = 2\sqrt3\,(t-\tfrac12)$$

$\hat U_3 = \dfrac{U_3}{\|U_3\|} = \dfrac{t^2-t+\frac16}{\|U_3\|}$, where $\|U_3\| = \sqrt{\langle U_3,U_3\rangle} = \sqrt{\int_0^1 (t^2-t+\frac16)^2\,dt}$

$$= \sqrt{\int_0^1 t^4 - 2t^3 + \tfrac43 t^2 - \tfrac13 t + \tfrac1{36}\,dt} = \sqrt{\frac15 - \frac12 + \frac49 - \frac16 + \frac1{36}} = \sqrt{\frac{-3}{10} + \frac49 - \frac5{36}}$$

$$= \sqrt{\frac49 - \frac{158}{360}} = \sqrt{\frac{160-158}{360}} = \sqrt{\frac{2}{360}} = \sqrt{\frac1{180}} = \frac{1}{\sqrt{180}}$$

Hence, $\hat U_3 = \dfrac{t^2-t+\frac16}{\frac{1}{\sqrt{180}}} = \sqrt{180}\,(t^2-t+\tfrac16) = \boxed{2\sqrt{45}\,(t^2-t+\tfrac16) = \hat U_3}$

therefore, the orthonormal basis is

$$\{1,\ 2\sqrt3\,(t-\tfrac12),\ 2\sqrt{45}\,(t^2-t+\tfrac16)\}$$

Section 5.2.

(7) (a) $V = \mathbb{R}^{n\times n}$ Define $\langle A, B\rangle = tr(AB)$. Show that this is a scalar product and that it is non-degenerate

Solution (i) scalar product $\Rightarrow$ Symmetric, Bilinear form.

Symmetry: we want to show that $\forall A, B \in \mathbb{R}^{n\times n}$ $\langle A,B\rangle = \langle B,A\rangle$

Pf: $\langle A, B\rangle = tr(AB)$ By definition of $\langle , \rangle$

By definition, $AB(i,k) = \sum_{j=1}^{n} A(i,j)B(j,k)$, But if we apply the function $tr(AB)$, we obtain

$$Tr(AB) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} A(i,j)B(j,i)\right).$$ Def of trace & Matrix multiplication

Interchanging the order of summation ... $$= \sum_{j=1}^{n}\sum_{i=1}^{n} B(j,i)A(i,j) = Tr(BA)$$

Hence, symmetry holds.

Bilinearity: we want to show that for any $s,t \in \mathbb{R}$ and for any $A,B,C \in \mathbb{R}^{n\times n}$: $\langle sA+tB, C\rangle = s\langle A,C\rangle + t\langle B,C\rangle$.

Pf: $\langle sA+tB, C\rangle = tr((sA+tB)\cdot C)$ ... Definition of $\langle , \rangle$

$= tr(s(AC)+t(BC))$ ... Property SP2 & SP3

$= \sum_{i=1}^{n}\left(s\sum_{j=1}^{n} A(i,j)C(j,i) + t\sum_{j=1}^{n} B(i,j)C(j,i)\right)$ Def of trace & matrix multiplication

We can separate summations ... $= \sum_{i=1}^{n} s\sum_{j=1}^{n} A(i,j)C(j,i) + \sum_{i=1}^{n} t\sum_{j=1}^{n} B(i,j)C(j,i)$

$= s\sum_{i=1}^{n}\sum_{j=1}^{n} A(i,j)C(j,i) + t\sum_{i=1}^{n}\sum_{j=1}^{n} B(i,j)C(j,i)$ ... Scalar multiplication

$= s\,tr(AC) + t\,tr(BC)$ ... Definition of Tr( )

$= s\langle A,C\rangle + t\langle B,C\rangle$ Definition of $\langle , \rangle$

From symmetry we obtain that $\langle A, sB+tC\rangle = s\langle A,B\rangle + t\langle A,C\rangle$

Hence, $\langle , \rangle$ is a bilinear form. It is also symmetric

thus, it is a scalar product

(ii) Show that this is a non-degenerate scalar product

Pf We want to show that given $A \in \mathbb{R}^{n \times n}$ if $\langle A, B \rangle = 0$ $\forall B \in \mathbb{R}^{n \times n}$ then $A = \Theta$

Assume that $\langle A, B \rangle = 0$ $\forall B \in \mathbb{R}^{n \times n}$. By definition,

$\langle A, B \rangle = tr(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} A(i,j) B(j,i) = 0$. The only way that this is true is if one of three cases hold:

(1) $A(i,j) = 0$ $\forall i,j$ (2) $B(j,i) = 0$ $\forall i,j$ or (3) Both (1) & (2) hold

If (1) holds, then $A = \Theta$ and we are done

(2) cannot be the case because we are assuming that $\langle A, B \rangle = 0$ $\forall B \in \mathbb{R}^{n \times n}$. Simply select $B \neq \Theta$ (e.g. $B = I$) to see that (2) is not true.

Because (2) is not true, (3) cannot be true

We are left with (1) which shows that ... is nondegenerate

5.2.7 (b) Show that the trace defines a positive definite scalar product on the space of real symmetric matrices

Pf: Let A be a real symmetric matrix. Then,

$$tr(AA) = \sum_{i=1}^{n} \sum_{j=1}^{n} A(i,j) A(j,i) \quad \ldots \text{Definition of trace}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} A(i,j)^2 \quad \ldots \text{By hypothesis } A \text{ is symmetric, so } A(i,j) = A(j,i)$$

Hence, if $A = \Theta$ then $A(i,j) = 0$ $\forall i,j$ and the trace is zero.

But, if $A \neq \Theta$ then $\exists (i,j)$ such that $A(i,j) \neq 0$. Furthermore $A(i,j)^2 > 0$ which will make the trace to be greater than zero, the sum of positive numbers, with one of them being greater than zero is greater than zero.

We can conclude that the trace defines a positive definite scalar product on the space of real symmetric matrices.

Enrique Areyan – M409 – Homework 3

(5.2.7) (c)(i) By work done on previous homework (exercise 2.1.6) we know that the dimension of the vector space of real $n \times n$ symmetric matrices is $\frac{n(n+1)}{2}$.

(ii) Let $W \subset V = \{A \mid tr(A) = 0\}$. To derive its dimension, consider the following linear map $L: \mathbb{R}^{n\times n}_{symmetric} \to \mathbb{R}$ by $L(A) = tr(A)$.

By the theorem of dimension of linear mappings we have that

$$Dim(\mathbb{R}^{n\times n}_{symmetric}) = Dim(Ker(L)) + Dim(Img(L))$$

$$\Rightarrow \quad \frac{n(n+1)}{2} - Dim(Img(L)) = Dim(Ker(L))$$

And $Ker(L) = \{A \mid tr(A) = 0\} = W$. We only need to know the dimension of the image of L to derive what we want.

But the $Img(L) = \mathbb{R}$, just take the matrix E such that $E(i,j) = 1$ if $i = j = 1$ and 0 otherwise. Cleary $E \in \mathbb{R}^{n\times n}_{symmetric}$ and $L(E) = 1$, thus $\{1\}$ is a subset of $Im(T)$ and generates all of $\mathbb{R}$. Hence, $Dim(Ker(L)) = dim(W) = \frac{n(n+1)}{2} - 1$

(iii) $W^{\perp} = \{B \in \mathbb{R}^{n\times n}_{symmetric} \mid \langle A, B \rangle = 0 \text{ and } tr(A) = 0\}$

By theorem $\mathbb{R}^{n\times n}_{symmetric} = \{W\}^{\perp} \oplus \langle\{W\}\rangle$. In particular, because the space $\mathbb{R}^{n\times n}_{symmetric}$ is finite

$$dim(\mathbb{R}^{n\times n}_{symmetric}) = dim(\{W\}^{\perp}) + dim(\langle\{W\}\rangle)$$

$$\Rightarrow \quad \frac{n(n+1)}{2} = dim(\{W\}^{\perp}) + \frac{n(n+1)}{2} - 1$$

$$\Rightarrow \quad dim(\{W\}^{\perp}) = 1$$

(5.2.8) Describe the orthogonal complement of the subspace of diagonal matrices. What is the dimension of this orthogonal complement?

<u>Solution</u>: Let $\mathbb{R}^{n\times n}$ denote the v.s of all square matrices.

Let $W \subset \mathbb{R}^{n\times n}$ be $W = \{A \in \mathbb{R}^{n\times n} \mid A \text{ is a diagonal matrix}\}$. W is a subspace (By hypothesis) Its orthogonal complement is

$$W^{\perp} = \{B \in \mathbb{R}^{n\times n} \mid \langle A, B\rangle = 0 \quad \forall A \in \mathbb{R}^{n\times n}\}$$

By def. of $\langle , \rangle$
$$\Rightarrow W^{\perp} = \{B \in \mathbb{R}^{n\times n} \mid tr(AB) = 0, \ \forall A \in \mathbb{R}^{n\times n}\}$$

To describe B, take the trace of B with the elementary matrices

$tr(E_{11} B) = 0 \Rightarrow b_{11} = 0$; $tr(E_{22} B) = 0 \Rightarrow b_{22} = 0$ ...

$tr(E_{nn} B) = 0 \Rightarrow b_{nn} = 0$, where $E_{ii} = 1$ in diagonal (i,i)

Hence, B is a matrix with 0 in its diagonal. The dimension of $W^{\perp}$ is:

$$\dim(\mathbb{R}^{n\times n}) = \dim(W) + \dim(W^{\perp}) \qquad \text{By theorem.}$$

We know from previous work that $\dim(W) = \dim(\text{diagonal matrices}) = n$. (Recall that a basis for W is $\{E_{ii} \mid E_{ii} = 1$ on diagonal entry (i,i) and 0 otherwise, for $1 \le i \le n\}$). We also know that $\dim(\mathbb{R}^{n\times n}) = n\times n$. Hence

$$n\times n = n + \dim(W^{\perp})$$
$$\Rightarrow n\times n - n = \dim(W^{\perp})$$
$$\boxed{n(n-1) = \dim(W^{\perp})}$$

We can easily see that a basis for $W^{\perp}$ consists of the same basis for the whole space <u>minus</u> the element of the basis that generate diagonal entries.

(5.2.9) Let V be a finite dimensional space over $\mathbb{R}$, with a pos. def scalar product. Let $\{v_1, \dots, v_m\}$ be a set of elements of V, of norm 1, and mutually perpendicular (i.e., $\langle v_i, v_j\rangle = 0$ if $i \neq j$). Assume that $\forall v \in V$ we have. $\|v\|^2 = \sum_{i=1}^{m} \langle v, v_i\rangle^2 = \langle v, v\rangle$ Show that $\{v_1, \dots, v_m\}$ is a basis of V.

<u>Pf</u>: to show that a set is a basis, we need to show two properties. (i) the set is independent, (ii) the set generates V

(i) Independence.

Pf by contradiction suppose $\{v_1, \dots, v_m\}$ is dependent then, without loss of generality, we can write one element as a linear combination of the others, i.e.,

$$v_k = \sum_{i \neq k} c_i v_i$$

By hypothesis, the scalar product $\langle , \rangle$ is pos definite. Hence,

$$\langle v_k, v_k \rangle > 0 \Rightarrow \langle \sum_{i \neq k} c_i v_i, v_k \rangle > 0.$$ Using linearity of $\langle , \rangle$: $\langle \sum_{i \neq k} c_i v_i, v_k \rangle = c_1 \langle v_1, v_k \rangle + \dots + c_{k-1} \langle v_{k-1}, v_k \rangle + c_{k+1} \langle v_{k+1}, v_k \rangle + \dots + c_m \langle v_m, v_k \rangle$

But, by hypothesis $\langle v_i, v_j \rangle = 0$ if $i \neq j$

$\Rightarrow c_1 \cdot 0 + \dots + c_{k-1} \cdot 0 + c_{k+1} \cdot 0 + \dots + c_m \cdot 0 = 0$, which contradicts the fact that $\langle \sum_{i \neq k} c_i v_i, v_k \rangle > 0$. therefore, $\{v_1, \dots, v_m\}$ must be an independent set.

(ii) Generates V. Pf: We just proved that $\{v_1, \dots, v_m\}$ is an independent set. We also know that the dimension of V is finite. Hence, the dimension of V is at least m. However, suppose that the dimension of V is greater than m. In particular, suppose that $\dim(V) = m+1$. then, a basis for V would be $\{v_1, \dots, v_m, v_{m+1}\}$. By theorem 2.1 we can convert the basis of the subspace of V $\{v_1, \dots, v_m\}$ to an orthogonal basis for V $\{v_1, \dots, v_m, v_{m+1}\}$ In other words, we can add v_{m+1} to $\{v_1, \dots, v_m\}$ to make an orthogonal basis for V.

But, given that $v_{m+1} \in V$, we can take:

$$0 < \langle v_{m+1}, v_{m+1} \rangle = \|v_{m+1}\|^2 = \sum_{i=1}^{m} \langle v_{m+1}, v_i \rangle^2$$

pos. def. scalar product def of norm

$$= \langle v_{m+1}, v_1 \rangle^2 + \dots + \langle v_{m+1}, v_m \rangle^2$$

Since v_{m+1} is orthogonal to every element in $\{v_1, \dots, v_m\}$

$$= 0^2 + \dots + 0^2 = 0 = \langle v_{m+1}, v_{m+1} \rangle$$

But, by definite pos. of the scalar product

$\boxed{\langle v_{m+1}, v_{m+1} \rangle > 0}$ Hence, we arrive at a contradiction

thus $\dim(V) = m$ which implies that $\{v_1, \dots, v_m\}$ is a basis of V.

Additional Exercise:

$B: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $B((v_1,v_2),(w_1,w_2)) = v_1w_2 + v_2w_1$

(i) Verify that B is a symmetric bilinear form:

symmetry: $B((v_1,v_2),(w_1,w_2)) = v_1w_2 + v_2w_1$ by definition of B

$= w_1v_2 + w_2v_1$ . . . . commutativity & Associativity of $\mathbb{R}$

$= B((w_1,w_2),(v_1,v_2))$

$\Rightarrow B((v_1,v_2),(w_1,w_2)) = B((w_1,w_2),(v_1,v_2))$ B is symmetric

(ii) Find a basis $\{b_1, b_2\}$ of $\mathbb{R}^2$ so that $B(b_1,b_1)=0$ and $B(b_2,b_2)=0$

Let $b_1=(1,0)$. then

$B(b_1,b_1) = B((1,0),(1,0)) = 1\,0 + 0.1 = 0$

Let $b_2 = (0,1)$ then

$B(b_2,b_2) = B((0,1),(0,1)) = 0.1 + 1.0 = 0$

Note that $\{b_1,b_2\} = \{\binom{1}{0},\binom{0}{1}\}$, the canonical basis for $\mathbb{R}^2$

But, B is not the trivial bilinear form, because $\exists\, a,b \in \mathbb{R}^2 \cdot B(a,b) \neq 0$

TAKE $B((2,1),(1,2)) = 2.2 + 1.1 = 4+1 = 5 \neq 0.$