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Section 5.4

(3)(i) Show that the bilinear maps of $\mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}$ form a v.s.

Solution: $\mathbb{B} = \{B \mid B \text{ is a bilinear form}, B: \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}\}$. We want to show that $\mathbb{B}$ is a v.s. with the operations:

'+': $\mathbb{B} \times \mathbb{B} \to \mathbb{B}$, defined by: If $B \in \mathbb{B}$ and $G \in \mathbb{B}$, then
$(B+G)(x,y) = B(x,y) + G(x,y)$, where $x \in \mathbb{K}^n$, $y \in \mathbb{K}^m$

'$\cdot$': $\mathbb{K} \times \mathbb{B} \to \mathbb{B}$, defined by: If $s \in \mathbb{K}$ and $B \in \mathbb{B}$, then
$(s \cdot B)(x,y) = sB(x,y)$, where $x \in \mathbb{K}^n$, $y \in \mathbb{K}^m$

But $\mathbb{B}$ is a subset of the v.s. of all functions. Hence, to prove that $\mathbb{B}$ is a v.s. we can check the three properties for a subspace:

(1) $\Theta \in \mathbb{B}$. This is true. Take $\Theta(x,y) = 0 \ \forall x \in \mathbb{K}^n \ \forall y \in \mathbb{K}^m$ then, it is trivial to see that Θ is bilinear: $\Theta(sx+s'x', y) = 0 = s\Theta(x,y) + s'\Theta(x',y) = s.0 + s'.0 = 0 = \Theta(x, sy+s'y')$.

Also, Let $B \in \mathbb{B}$. then, $(\Theta + B)(x,y) = \Theta(x,y) + B(x,y)$ Def '+'
$= 0 + B(x,y)$ Def Θ
$= B(x,y)$ identity element in $\mathbb{K}$

the other combination is also true:
$(B+\Theta)(x,y) = B(x,y) + \Theta(x,y) = B(x,y) + 0 = B(x,y)$.

(2) Given $B, G \in \mathbb{B}$ $B+G \in \mathbb{B}$.

We want to show that the sum of bilinear maps of $\mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}$ results in another bilinear form also from $\mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}$.

Let $B, G \in \mathbb{B}$: $(B+G)(x,y) = B(x,y) + G(x,y)$ Def of '+'

Let $H(x,y) = B(x,y) + G(x,y)$. Is H bilinear?

$H(sx+s'x', y) = B(sx+s'x', y) + G(sx+s'x', y)$

By linearity of B & G $= (sB(x,y) + s'B(x',y)) + (sG(x,y) + s'G(x',y))$

Associativity & Commutativity of $\mathbb{K}$ $= (sB(x,y) + sG(x,y)) + (s'B(x',y) + s'G(x',y))$

Distributivity of $\mathbb{K}$ $= s(B(x,y) + G(x,y)) + s'(B(x',y) + G(x',y))$

$= sH(x,y) + s'H(x',y)$.

Hence, $H(sx+s'x', y) = sH(x,y) + s'H(x',y)$. By the same arguments, is true that $H(x, sy+s'y') = sH(x,y) + s'H(x,y')$. Thus, H is bilinear $\Rightarrow H \in \mathbb{B}$

(3) Given $s \in \mathbb{K}$ and $B \in \mathbb{B}$: $sB \in \mathbb{B}$.

Let $t \in \mathbb{K}$ and $G \in \mathbb{B}$: TAKE $(t \cdot G)(x,y) = H$. We want to show that $H \in \mathbb{B}$, i.e, H is a bilinear form of $\mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$

$$\begin{aligned} H(sx+s'x', y) &= t\,G(sx+s'x', y) && \text{By definition of } H \\ &= t(sG(x,y)+s'G(x',y)) && \text{By linearity of } G \\ &= st\,G(x,y)+s't\,G(x',y)) && \text{Distributivity \& associativity of } \mathbb{K} \\ &= sH(x,y)+s'H(x',y) && \text{Def of } H. \end{aligned}$$

By the same arguments, $H(x, sy+s'y') = sH(x,y)+s'H(x,y')$

Hence, $H \in \mathbb{B}$

$\Rightarrow$ $\mathbb{B}$ is a v.s. over $\mathbb{K}$.

(ii) More generally, let $Bil(U \times V, W)$ be the set of bilinear maps of $U \times V$ into W. Show that $Bil(U\times V, W)$ is a v.s.

Solution: Like before, this is a subset of all functions. Hence, we need only to check the following three properties:

(1) $\theta \in Bil(U\times V, W)$. the additive identity is $\theta(x,y) = \theta_W$, where $x \in U$, $y \in V$ and $\underbrace{\theta_W}_{\text{identity in } W} \in W$. Just as before, this is a bilinear form:

$\theta(sx+s'x', y) = \theta_W = \theta_W + \theta_W = s\theta(x,y) + s'\theta(x',y)$. Hence, $\theta \in Bil$.

Also, let $B \in Bil(U\times V, W)$, then $(B+\theta)(x,y) = B(x,y)+\theta(x,y)$ $= B(x,y) + \theta_W = B(x,y)$. Furthermore, $(\theta + B)(x,y) = \theta(x,y) + B(x,y)$ $= \theta_W + B(x,y) = B(x,y)$. Hence, $\theta(x,y) = \theta_W$ $\forall x,y \in U\times V$ is the identity and it is inside of $Bil(U\times V, W)$

(2) We want to show that given $B, G \in Bil(U\times V, W)$, then $B+G \in Bil(U\times V, W)$. In words, we want to show that the sum of Bilinear forms from $U\times V$ to W is also a bilinear form from $U\times V$ to W.

Let $C, D \in Bil(U\times V, W)$. TAKE the sum:

$$E = (C+D)(x,y) = C(x,y)+D(x,y) \qquad \text{By def of '+'}$$

Check that $E: U\times V \to W$ is a bilinear form:

$$\begin{aligned} E(sx+s'x', y) &= C(sx+s'x', y) + D(sx+s'x', y) && \text{By def of } E \\ &= (sC(x,y)+s'C(x',y)) + (sD(x,y)+s'D(x',y)) && \text{By linearity in the first parameter of } C \,\&\, D \\ &= (sC(x,y)+sD(x,y)) + (s'C(x',y)+s'D(x',y)) && \text{Commutativity and associativity of the space } W. \end{aligned}$$

$= s(C(x,y)+D(x,y)) + s'(C(x',y)+D(x',y))$ . . . (VS5) of space W

$= s\,E(x,y) + s'E(x',y)$ . . . Def of E

Hence, E is linear in the first parameter. To prove that is linear in the second parameter we do the same thing (I ommit justifications to make it cleaner): $E(x, sy+s'y') = C(x, sy+s'y') + D(x, sy+s'y')$
$= (sC(x,y) + s'C(x,y')) + (sD(x,y) + s'D(x,y'))$
$= (sC(x,y) + sD(x,y)) + (s'C(x,y') + s'D(x,y'))$
$= s(C(x,y) + D(x,y)) + s'(C(x,y') + D(x,y'))$
$= s\,E(x,y) + s'E(x,y')$. Hence, E is linear in the second parameter.
It follows that E is bilinear, showing that addition is closed on $Bil(U\times V, W)$.

(3) Given $s\in \mathbb{K}$, where $\mathbb{K}$ is the field associated with W. and $B\in Bil$, we want to show that $s\cdot B\in Bil$.
$(s.B)(x,y) = s(B(x,y))$ But, by definition B is bilinear, so sB will also be a bilinear form. Hence $s.B\in Bil$.

Because (1), (2) & (3) hold, $Bil(U\times V, W)$ is a VS.

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(4) Show that the association $A\mapsto g_A$ is an isomorphism between the space of $m\times n$ matrices, and the space of bilinear maps of $\mathbb{K}^m\times \mathbb{K}^n$ into $\mathbb{K}$.

Solution: By definition, a linear map $F: U\to V$ which has an inverse $G: V\to U$ is called an isomorphism.
In this case we want to prove that $F: A\mapsto g_A$ is an isomorphism, i.e, that it is a linear map which has an inverse.
We must first check linearity, i.e, given $B, C\in A$, and $s, s'\in\mathbb{K}$ we want to show that

$$F(sB+s'C) \stackrel{?}{=} sF(B) + s'F(C)$$

By theorem 4.1 $\Rightarrow$ $F(sB+s'C) = g_{sB+s'C}(X,Y) = {}^tX(sB+s'C)Y$. By basic properties of matrices $= s\,{}^tXBY + s'\,{}^tXCY = s\,g_B(X,Y) + s'g_C(X,Y)$
$= sF(B) + s'F(C)$.

Hence, F is linear.
Now we must check that F has an inverse or is invertible

From previous work (corollary 4.4), we know that a linear map that is injective and surjective has an inverse. Both injectivity and surjectivity of F follow from theorem 4.1. F is injective because given matrices A, B if $F(A)=F(B) \Rightarrow A=B$ by uniqueness of the matrix associated with a bilinear map. Likewise, for every bilinear map, there exists a matrix such that $F(A)=g_A(x,y)$. Hence, F is injective & surjective which implies that it has an inverse.

In conclusion, F is an invertible linear map, i.e. an isomorphism.

(5) (a) $(x_1\ x_2)\begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_1\ x_2)\begin{pmatrix} 2y_1-3y_2 \\ 4y_1+y_2 \end{pmatrix} = x_1(2y_1-3y_2)+x_2(4y_1+y_2)$

$= 2x_1y_1-3x_1y_2+4x_2y_1+x_2y_2$

(b) $(x_1\ x_2)\begin{pmatrix} 4 & 1 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_1\ x_2)\begin{pmatrix} 4y_1+y_2 \\ -2y_1+5y_2 \end{pmatrix} = x_1(4y_1+y_2)+x_2(-2y_1+5y_2)$

$= 4x_1y_1+x_1y_2-2x_2y_1+5x_2y_2$

(c) $(x_1\ x_2)\begin{pmatrix} -5 & 2 \\ \pi & 7 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_1\ x_2)\begin{pmatrix} -5y_1+2y_2 \\ \pi y_1+7y_2 \end{pmatrix} = x_1(-5y_1+2y_2)+x_2(\pi y_1+7y_2)$

$= -5x_1y_1+2x_1y_2+\pi x_2y_1+7x_2y_2$

(d) $(x_1\ x_2\ x_3)\begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 4 \\ 2 & 5 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = (x_1\ x_2\ x_3)\begin{pmatrix} y_1+2y_2-y_3 \\ -3y_1+y_2+4y_3 \\ 2y_1+5y_2-y_3 \end{pmatrix}$

$= x_1y_1+2x_1y_2-x_1y_3-3x_2y_1+x_2y_2+4x_3y_3+2x_3y_1+5x_3y_2-x_3y_3$

(e) $(x_1\ x_2\ x_3)\begin{pmatrix} -4 & 2 & 1 \\ 3 & 1 & 1 \\ 2 & 5 & 7 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = (x_1\ x_2\ x_3)\begin{pmatrix} -4y_1+2y_2+y_3 \\ 3y_1+y_2+y_3 \\ 2y_1+5y_2+7y_3 \end{pmatrix}$

$= -4x_1y_1+2x_1y_2+x_1y_3+3x_2y_1+x_2y_2+x_2y_3+2x_3y_1+5x_3y_2+7x_3y_3$

(f) $(x_1\ x_2\ x_3)\begin{pmatrix} -1/2 & 2 & -5 \\ 1 & 2/3 & 4 \\ -1 & 0 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = (x_1\ x_2\ x_3)\begin{pmatrix} -\frac{1}{2}y_1+2y_2-5y_3 \\ y_1+2/3y_2+4y_3 \\ -y_1+0y_2+3y_3 \end{pmatrix}$

$= -\frac{1}{2}x_1y_1+2x_1y_2-5x_1y_3+x_2y_1+\frac{2}{3}x_2y_2+4x_2y_3-x_3y_1+3x_3y_3$

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(6) the vectors are e_1 and e_2, because:

$$g(e_1,e_2) = g((1,0,0),(0,1,0)) = 2 = \text{the } C_{12} \text{ element of } C$$

But, $g(e_2,e_1) = g((0,1,0),(1,0,0)) = -1 = $ the C_{21} element of C.

hence, $g(e_1,e_2) \neq g(e_2,e_1)$. this follows from the fact that C is not a symmetric matrix, hence the bilinear form associated with C is not symmetric.

Section 5.5.

(1) (a) Let $U_1 = A$. Note that $A \cdot A = 1+2+1 = 4 \neq 0$, so we can continue

Let $U_2 = B - P_{\langle A \rangle}(B) = (1,-1,2) - \frac{A \cdot B}{A \cdot A} A$

$$= (1,-1,2) - \frac{1}{4}(1,1,1) = \frac{1}{4}(3,-5,7)$$

Hence, one orthogonal basis for the subspace is $\{U_1,U_2\} = \{(1,1,1), \frac{1}{4}(3,-5,7)\}$

We can check that indeed this is orthogonal, i.e.,

$$U_1 \cdot U_2 = (1,1,1) \cdot \frac{1}{4}(3,-5,7) = \frac{3}{4} - \frac{10}{4} + \frac{7}{4} = 0 \quad \checkmark$$

(1) (b) Note that $\{A,B\}$ is already an orthogonal basis:

$$A \cdot B = -1 - 3(-1 \cdot 1) + 3 + (-4) - 4 - (-1 \cdot 3) = -1 + 3 + 3 - 4 - 4 + 3 = 8 - 8 = 0$$

Section 5.7

(1) By definition, f is a quadratic form if $f(v) = \langle v,v \rangle$ $\forall v \in V$, where $\langle , \rangle$ is a symmetric bilinear form. Using the definition of g:

$$g(v,v) = f(v+v) - f(v) - f(v) = f(2v) - 2f(v) =$$ By property of f (hypothesis)

$$= 4f(v) - 2f(v)$$
$$= 2f(v)$$

$\Rightarrow 2f(v) = g(v,v) \Rightarrow f(v) = \frac{1}{2} g(v,v)$.

Note that By hypothesis g is bilinear. It is also symmetric:

$$g(v,w) = f(v+w) - f(v) - f(w) = f(w+v) - f(w) - f(v) = g(w,v).$$

Hence, f is a quadratic form. the bilinear form from which it comes from is $\frac{1}{2} g(v,w)$.

Suppose that f also comes from another bilinear form h then, by formula given in the book

$$h(v,w) = \frac{1}{2}[h(v+w,v+w) - h(v,v) - h(w,w)] =$$
$$= \frac{1}{2}[f(v+w) - f(v) - f(w)] = g(v,w)$$

$\Rightarrow h(v,w) = g(v,w)$, so the underlying bilinear form is unique.

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(3) What is the associated matrix of the quadratic form
$$f(X) = x^2 - 3xy + 4y^2 \quad \text{if } {}^tX = (x,y,z)?$$

Solution:
$$[x,y,z]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}[x\ y\ z] = x^2 - 3xy + 4y^2$$

$$\Rightarrow [x\ y\ z]\begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix} = x^2 - 3xy + 4y^2$$

$$a_{11}x^2 + a_{12}xy + a_{13}xz + a_{21}xy + a_{22}y^2 + a_{23}yz + a_{31}xz + a_{32}yz + a_{33}z^2 = x^2 - 3xy + 4y^2$$

But we know that a quadratic form is determined only in terms of a symmetric bilinear form. Hence, the matrix must be symmetric

$$a_{11}x^2 + 2a_{12}xy + 2a_{13}xz + a_{22}y^2 + 2a_{23}yz + a_{33}z^2 = x^2 - 3xy + 4y^2$$

$\Rightarrow a_{11} = 1$, $2a_{12} = -3 \Rightarrow a_{12} = -\frac{3}{2}$, $a_{13} = a_{23} = a_{33} = 0$, $a_{22} = 4$

Hence, the matrix is: $\begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ -3/2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

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(4) Show that if $f_1(v) = g_1(v,v)$ and $f_2(v) = g_2(v,v) \Rightarrow (f_1+f_2)(v) = (g_1+g_2)(v,v)$.

Pf:

$(f_1+f_2)(v) = f_1(v) + f_2(v)$ . . . Def (+)

$= g_1(v,v) + g_2(v,v)$ . . . . . Hypothesis

$= (g_1+g_2)(v,v)$ . . . . . Def (+)

Proving what we wanted to prove.

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Section 5.8

(1) (a) $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ the vector $e_1 = (1, 0)$ form part of some orthogonal basis with respect to the product determined by this matrix. Hence, $e_1 A e_1 = 1$, so we know that at least one vector of an orthogonal basis is positive.

Likewise, $e_2 = (0,1)$ form part of some orthogonal basis (not necessarily the same basis as before). Hence $e_2 A e_2 = -1$. $\Rightarrow$ at least one negative vector. But, the basis has exactly two elements. the signature is $(0,1,1)$.

(b) Using the same reasoning as before, $e_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} e_1 = 1 \Rightarrow$ at least one positive. However, we need to complete this basis:

$U_1 = e_1$, $U_2 = e_2 - P_{\langle U_1 \rangle} e_2 = e_2 - \frac{e_2 \cdot e_1}{e_1 \cdot e_1} e_1$,

$e_2 \cdot e_1 = e_2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} e_1 = 1$; $e_1 \cdot e_1 = 1$

$\Rightarrow U_2 = (0,1) - (1,0) = (-1,1)$. we can check that $\{U_1, U_2\}$ is an orthogonal basis $U_1 \cdot U_2 = (1,0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (1\ 0) \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$.

Compute, $U_1 \cdot U_1 = e_1 \cdot e_1 = 1$ and $U_2 \cdot U_2 = (-1\ 1) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
$= (-1\ 1) \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$.

the signature is $(1,1,0)$.

(c) $\begin{pmatrix} 1 & -3 \\ -3 & 2 \end{pmatrix}$. Same as before: $e_1 A e_1 = 1$.

Let $U_1 = e_1$. $U_2 = e_2 - P_{\langle U_1 \rangle} e_2 = e_2 - \frac{e_2 \cdot U_1}{U_1 \cdot U_1} U_1 = (0,1) - \frac{-3}{1}(1,0)$

$U_2 = (3,1)$

Check: $U_1 \cdot U_2 = (1,0) \begin{pmatrix} 1 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = (1,0) \begin{pmatrix} 0 \\ -7 \end{pmatrix} = 0$.

Compute $U_1 \cdot U_1 = e_1 A e_1 = 1$ and

$U_2 \cdot U_2 = (3,1) \begin{pmatrix} 1 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = (3,1) \begin{pmatrix} 0 \\ -7 \end{pmatrix} = -7$.

The signature is $(0,1,1)$

5.8.3: (a) $A = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$

Let $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be a basis for the vector space of all 2x2 symmetric matrices. then (x, y, z) are the coordinates of A w.r.t. B, i.e.

$$A = \begin{pmatrix} x & y \\ y & z \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) $W \subseteq V = \{A \mid Tr(A) = 0\}$. Show that for $A \in W$ and $A \neq 0$, we have $f(A) < 0$, where $f(A) = xz - y^2$

Solution: Let $A \in W$. By definition $tr(A) = 0$. this means that $x + z = 0 \Rightarrow x = -z$. $A \neq \theta$, hence $x \neq 0$ or $y \neq 0$ or $z \neq 0$

Also, $f(A) = xz - y^2$. If $A \in W$, then

$$= -zz - y^2$$
$$= -z^2 - y^2 = -x^2 - y^2 = (*)$$

the value of (*) depends on various cases:

If $y \neq 0$ then either

(1) $x \neq 0$ and $y \neq 0$, in which case $(*) < 0$ because, $-x^2 < 0$ and $-y^2 < 0$

or

(2) $x = y = 0$, in which case $(*) < 0$ because $0 - y^2 = -y^2 < 0$

If $y = 0$ then $x \neq 0$ and $z \neq 0$ in which case $(*) < 0$ because $-x^2 - 0 = -x^2 < 0$.

Hence, in all possible cases, $f(A) < 0$, which means that f is a negative definite quadratic form.

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Section 5.2:

(6) Find an orthonormal basis for the subspace of $\mathbb{C}^3$ generated by the following vectors:

(a) $(1,i,0)$ and $(1,1,1)$

Let $u_1=(1,i,0)$ and $v_2=(1,1,1)$.

$$u_2 = v_2 - P_{\langle u_1\rangle}^{v_2} = (1,1,1) - \frac{v_2\cdot u_1}{u_1\cdot v_1}u_1 \quad \text{, where}$$

$u_1\cdot u_1 = (1,i,0)\cdot(1,i,0) = 1-i^2 = 1-(-1) = 2$

$v_2\cdot u_1 = (1,1,1)\cdot(1,i,0) = 1+(-i)+0 = 1-i$. Hence,

$$u_2 = (1,1,1) - \frac{1-i}{2}(1,i,0) = (1,1,1) - \frac{1}{2}(1-i,1+i,0)$$
$$= (1-\tfrac{1}{2}(i-1), 1-\tfrac{1}{2}(i+1), 1-\tfrac{1}{2}(0))$$
$$= \tfrac{1}{2}(2-(1-i), 2-(i+1), 2)$$
$$= \tfrac{1}{2}(1+i, 1-i, 2)$$

Hence, $\{u_1,u_2\} = \{(1,i,0), \tfrac{1}{2}(1+i,1-i,2)\}$ is an orthogonal basis.

An orthonormal basis $\{\hat{u}_1,\hat{u}_2\}$ is,

$\hat{u}_1 = \frac{u_1}{\|u_1\|}$, $\|u_1\| = \sqrt{u_1\cdot u_1} = \sqrt{2} \Rightarrow \hat{u}_1 = \frac{1}{\sqrt{2}}(1,i,0)$

$\hat{u}_2 = \frac{u_2}{\|u_2\|}$, $\|u_2\| = \sqrt{u_2\cdot u_2} = \sqrt{\frac{1}{2}((1+i)(1-i)+(1-i)(1+i)+4)}$
$$= \sqrt{\tfrac{1}{2}(2((1+i)(1-i))+4)} = \sqrt{\tfrac{1}{2}(2\cdot 2+4)} = \sqrt{\tfrac{8}{2}} = 2$$

$\Rightarrow \hat{u}_2 = \frac{1}{4}(1+i,1-i,2)$.

(b) $(1,-1,-i)$ and $(i,1,2)$

Let $u_1=(1,-1,-i)$ and $v_2=(i,1,2)$

$$u_2 = v_2 - P_{\langle u_1\rangle}^{v_2} = (i,1,2) - \frac{v_2\cdot u_1}{u_1\cdot u_1}u_1, \quad \text{where}$$

$u_1\cdot u_1 = (1,-1,-i)\cdot(1,-1,-i) = 1+1+(-i\bar{i}) = 3$

$v_2\cdot u_1 = (i,1,2)\cdot(1,-1,-i) = i-1+2i = 3i-1$. Hence,

$$u_2 = (i,1,2) - \frac{3i-1}{3}(1,-1,-i) = \left(i-\frac{3i-1}{3}, 1+\frac{3i-1}{3}, 2+\frac{(3i-1)i}{3}\right)$$
$$= \left(\frac{1}{3}, \frac{3i+2}{3}, \frac{3-i}{3}\right) = \frac{1}{3}(1,3i+2,3-i)$$

$\{v_1, v_2\} = \{(1,-1,-i), \frac{1}{3}(1, 3i+2, 3-i)\}$ form an orthogonal basis.

An orthonormal basis $\{\hat{v}_1, \hat{v}_2\}$ is

$$\hat{v}_1 = \frac{v_1}{\|v_1\|}, \quad \|v_1\| = \sqrt{v_1 \cdot v_1} = \sqrt{3} \Rightarrow \hat{v}_1 = \frac{1}{\sqrt{3}}(1,-1,-i)$$

$$\hat{v}_2 = \frac{v_2}{\|v_2\|}, \quad \|v_2\| = \sqrt{v_2 \cdot v_2} = \sqrt{\frac{1}{9} + \left(i+\frac{2}{3}\right)\left(-i+\frac{2}{3}\right) + \left(1-\frac{i}{3}\right)\left(1+\frac{i}{3}\right)}$$

$$= \sqrt{\frac{1}{9} + \left(1+\cancel{\frac{2}{3}i}-\cancel{\frac{2}{3}i}+\frac{4}{9}\right) + \left(1+\cancel{\frac{i}{3}}-\cancel{\frac{i}{3}}+\frac{1}{9}\right)} = \sqrt{\frac{1}{9}+\frac{13}{9}+\frac{10}{9}} = \sqrt{\frac{24}{9}} =$$

$$\Rightarrow \hat{v}_2 = \frac{\frac{1}{3}}{\frac{\sqrt{24}}{3}}(1, 3i+2, 3-i) = \frac{1}{\sqrt{24}}(1, 3i+2, 3-i)$$

---

section 5.8

(2) By theorem 5.1, we know that there exists an orthogonal basis for V, call it $\{v_1, \dots, v_n\}$. By Sylvester's Law of Inertia, we know that regardless of the choosen basis, we can define the signature of V as $(\#\{v_i \mid B(v_i,v_i)=0\}, \#\{v_i \mid B(v_i,v_i)>0\}, \#\{v_i \mid B(v_i,v_i)<0\})$.

Arrange the orthogonal basis of V such that

$\langle v_i, v_i\rangle = 0$ if $1 \le i \le r$

$\langle v_i, v_i\rangle > 0$ if $r+1 \le i \le s$

$\langle v_i, v_i\rangle < 0$ if $s+1 \le i \le n$.

By theorem 8.1, $\dim(V_0) = r$. Also, by partioning the basis as before, $\dim(V^+) = s-r$ and $\dim(V^-) = n-s$. Hence

$\dim(V) = n = \dim(V_0) + \dim(V^+) + \dim(V^-) = r+s-r+n-s = n$.

$\langle\{v_1,\dots,v_r\}\rangle = V_0$ and $\langle\{v_{r+1},\dots,v_s\}\rangle = V^+$ and $\langle\{v_{s+1},\dots,v_n\}\rangle = V^-$, hence

$$V = V_0 \oplus V^+ \oplus V^-.$$

By sylvester theorem, the dimension of V^+ is the same regardless of the basis as so is the dimension of V^-. (the signature is the same).

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section 5.6

(2) We know that the ordinary dot product of vectors in $\mathbb{C}^n$ is pos. def. $\Rightarrow \mathbb{C}^n = \{w\}^\perp \oplus \langle\{w\}\rangle \Rightarrow \dim(\mathbb{C}^n) = \dim(\{w\}^\perp) + \dim(\langle\{w\}\rangle)$

In this case we can think of $w = \langle\{x,y\}\rangle$ where $x, y \in \mathbb{C}^n$.

Hence, $\dim(\mathbb{C}^n) = \dim(\{x,y\}^\perp) + \dim(\langle\{x,y\}\rangle)$. By definition, $\dim(\mathbb{C}^n) = n$ and $\dim(\langle\{x,y\}\rangle) = 2$. therefore,

$$\boxed{\dim(\{x,y\}^\perp) = n-2}$$

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Section 6.3

(1) Compute the following determinants.

(a) $\begin{vmatrix} 2 & 1 & 2 \\ 0 & 3 & -1 \\ 4 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & -1 \\ 4 & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 3 \\ 4 & 1 \end{vmatrix}$

$= 2(3+1) - 1(+4) + 2(-12) = 8 - 4 - 24 = \underline{-20}$

(b) $\begin{vmatrix} 3 & -1 & 5 \\ -1 & 2 & 1 \\ -2 & 4 & 3 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} + 5\begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix}$

$= 3((6-4)) + 1(-3+2) + 5(-4+4) = 6 - 1 = \underline{5}$

(c) $\begin{vmatrix} 2 & 4 & 3 \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix}$

$= 2(3) - 4(-1) + 3(-2) = 6 + 4 - 6 = \underline{4}$

(d) $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 7 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 0 & 7 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} = \underline{5}$

(e) $\begin{vmatrix} -1 & 5 & 3 \\ 4 & 0 & 0 \\ 2 & 7 & 8 \end{vmatrix} = -1\begin{vmatrix} 0 & 0 \\ 7 & 8 \end{vmatrix} - 5\begin{vmatrix} 4 & 0 \\ 2 & 8 \end{vmatrix} + 3\begin{vmatrix} 4 & 0 \\ 2 & 7 \end{vmatrix}$

$= -1(0) - 5(32) + 3(28) = -160 + 84 = \underline{-76}$

(f) $\begin{vmatrix} 3 & 1 & 2 \\ 4 & 5 & 1 \\ -1 & 2 & -3 \end{vmatrix} = 3\begin{vmatrix} 5 & 1 \\ 2 & -3 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 \\ -1 & -3 \end{vmatrix} + 2\begin{vmatrix} 4 & 5 \\ -1 & 2 \end{vmatrix} =$

$= 3(-15-2) - 1(-12+1) + 2(8+5) = -51 + 11 + 26 = \underline{-14}$

(3) In general, the determinant of a diagonal matrix is:

$\det\left(\begin{vmatrix} a_{11} & & & 0 \\ & a_{12} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{vmatrix}\right) = a_{11} \det\left(\begin{vmatrix} a_{12} & & 0 \\ & a_{13} & \\ 0 & & a_{nn} \end{vmatrix}\right) + 0 \cdot \det() + \cdots + 0 \cdot \det()$

$= a_{11}\left(a_{12} \cdot \det\begin{vmatrix} a_{13} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{vmatrix}\right) + 0 \cdot \det() + \cdots + 0 \det() = \cdots =$

$= \boxed{a_{11} a_{12} \cdots a_{nn}}$, i.e, the product of its diagonal terms.

Additional Exercises.

THEOREM: Let $\mathbb{K}$ be a field. For each positive integer n, there exists a unique function $D: \underbrace{\mathbb{K}^n \times \cdots \times \mathbb{K}^n}_{n \text{ times}} \to \mathbb{K}$, that satisfies the following properties:

I. $D(v_1, \dots, cv_j, \dots, v_n) = c\, D(v_1, \dots, v_j, \dots, v_n)$

II. $D(v_1, \dots, v_j + w, \dots, v_n) = D(v_1, \dots, v_j, \dots, v_n) + D(v_1, \dots, w, \dots, v_n)$

III. $D(v_1, \dots, v_j, v_{j+1}, \dots, v_n) = -D(v_1, \dots, v_{j+1}, v_j, \dots, v_n)$

IV. $D(e_1, \dots, e_n) = 1$.

(1) Verify II in the inductive step.

Solution: We want to prove that $\boxed{D(v_1, \dots, v_j + w, \dots, v_{n+1}) = D(v_1, \dots, v_j, \dots, v_{n+1}) + D(v_1, \dots, w, \dots, v_{n+1})}$ Assuming the inductive hypothesis on n.

Pf. By definition of D:

$$D(v_1, \dots, v_j + w, \dots, v_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} v_i^1 D(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_j + w, \dots, v_{n+1})$$

We can split the summation into three pieces as follow:

$$= \sum_{i=1}^{j-1} (-1)^{i+1} v_i^1 D(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_j + w, \dots, v_{n+1}) + (-1)^{j+1} (v_j^1 + w^1)\, D(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_{n+1}) + \sum_{i=j+1}^{n+1} (-1)^{i+1} v_i^1 D(v_1, \dots, v_j + w, \dots, v_{i-1}, v_{i+1}, \dots, v_{n+1}) =$$

Using the inductive hypothesis on the first and third elements of this summation, and distributing $v_j^1 + w^1$ over D, which is a valid operation of the field $\mathbb{K}$, we obtain:

$$= \sum_{i=1}^{j-1} (-1)^{i+1} v_i^1 \Big(D(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_j, \dots, v_{n+1}) + D(v_1, \dots, v_{i-1}, v_{i+1}, \dots, w, \dots, v_{n+1}) \Big) + \Big((-1)^{j+1} v_j^1 D(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_{n+1}) + (-1)^{j+1} w^1 D(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_{n+1}) \Big) + \sum_{i=j+1}^{n+1} (-1)^{i+1} v_i^1 \Big(D(v_1, \dots, v_j, \dots, v_{i-1}, v_{i+1}, \dots, v_{n+1}) + D(v_1, \dots, w, \dots, v_{i-1}, v_{i+1}, \dots, v_{n+1}) \Big) =$$

Distributing and associating:

$$= \Big(\Big(\sum_{i=1}^{j-1} (-1)^{i+1} v_i^1 D(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_j, \dots, v_{n+1}) \Big) + (-1)^{j+1} v_j^1 D(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_{n+1}) + \sum_{i=j+1}^{n+1} (-1)^{i+1} v_i^1 D(v_1, \dots, v_j, \dots, v_{i-1}, v_{i+1}, \dots, v_{n+1}) \Big) + \Big(\Big(\sum_{i=1}^{j-1} (-1)^{i+1} v_i^1 D(v_1, \dots, v_{i-1}, v_{i+1}, \dots, w, \dots, v_{n+1}) \Big) + (-1)^{j+1} w^1 D(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_{n+1}) + \sum_{i=j+1}^{n+1} (-1)^{i+1} v_i^1 D(v_1, \dots, w, \dots, v_{i-1}, v_{i+1}, \dots, v_{n+1}) \Big) =$$

By definition of D $= D(v_1, \dots, v_j, \dots, v_{n+1}) + D(v_1, \dots, w, \dots, v_{n+1})$ //

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(2) Verify III in the inductive step.

Solution: We want to prove $D(v_1,\dots,v_j,v_{j+1},\dots,v_{n+1}) = -D(v_1,\dots,v_{j+1},v_j,\dots,v_{n+1})$

Pf: By definition of D:

$$D(v_1,\dots,v_j,v_{j+1},\dots,v_{n+1}) = \sum_{i=1}^{n+1}(-1)^{i+1}v_i^1 D(v_1,\dots,v_{i-1},v_{i+1},\dots,v_j,v_{j+1},\dots,v_{n+1}) =$$

which we can decompose as:

$$= \sum_{i=1}^{j-1}(-1)^{i+1}v_i^1 D(v_1,\dots,v_{i-1},v_{i+1},\dots,v_j,v_{j+1},\dots,v_{n+1}) +$$

$$(-1)^{j+1}v_j^1 D(v_1,\dots,v_{j-1},v_{j+1},\dots,v_{n+1}) +$$

$$\sum_{i=j+1}^{n+1} v_i^1 D(v_1,\dots,v_j,v_{j+1},\dots,v_{i-1},v_{i+1},\dots,v_{n+1}) =$$

Using the inductive hypothesis.

$$= -\Big(\sum_{i=1}^{j-1}(-1)^{i+1}v_i^1 D(v_1,\dots,v_{i-1},v_{i+1},\dots,v_{j+1},v_j,\dots,v_{n+1}) +$$

$$(-1)^j v_j^1 D(v_1,\dots,v_{j-1},v_{j+1},\dots,v_{n+1}) +$$

$$\sum_{i=j+1}^{n+1} v_i^1 D(v_1,\dots,v_{j+1},v_j,\dots,v_{i-1},v_{i+1},\dots,v_{n+1})\Big) =$$

By definition of D

$$= -D(v_1,\dots,v_{j+1},v_j,\dots,v_{n+1})$$

(3) Show that if $v_i = v_j$ for some i not equal to j then $D(v_1,\dots,v_n)=0$

Pf: If $v_i = v_j$ for some contiguous (i,j), i.e., $i=j+1$ OR, $v_i=v_j$ where either $i=1$ and $j=n$ OR $i=n$ and $j=1$, then we can directly use property III of D to conclude that:

$$D(v_1,\dots,v_n) = -D(v_1,\dots,v_n) \Rightarrow 2D(v_1,\dots,v_n)=0 \Rightarrow D(v_1,\dots,v_n)=0$$

If $v_i=v_j$ is such that i and j are not contiguous, the claim is that $D(v_1,\dots,v_n)=0$ still. To prove this, it is easier to use the following notation:

let $f\in Aut(\{1,\dots,n\})$ then $D(v_{f(1)},\dots,v_{f(n)}) = \varepsilon(f)\cdot D(v_1,\dots,v_n)$,

We know that a permutation $f\in Aut(\{1,\dots,n\})$ can be written as product of transpositions, i.e., $f=\tau_1\circ\dots\circ\tau_k$.

We need to deal with two cases:

(1) $v_i = v_j$ where i and j are separated by an even number of elements. In this case, we will need a odd permutation such that $f(i)=j$ and $f(j)=i$ and $f(k)=k$ $\forall i \notin \{i,j\}$. this is an odd permutation because we have to swap an even number of elements plus one more swap for i and j.

(2) $v_i = v_j$ where i and j are separated by an odd number of elements. In this case we also need an odd permutation such that $f(i)=j-1$ and $f(j-1)=i$ and $f(k)=k$ $\forall k \notin \{i,j-1\}$.

$\Rightarrow$ in both cases $D(v_1,..,v_i,..,v_j,..,v_n) = (-1)D(v_1,..,v_i,v_j,..,v_n)$, where $v_i=v_j \Rightarrow 2D(v_1,..,v_i,v_j,..,v_n)=0$

$\Rightarrow D(v_1,\dots,v_n)=0$. the difference is that in (1) we use one more swap to interchange v_i with v_j whereas in (2) we don't need that last swap. Hence, we can always use an odd permutation to arrive to our conclusion.

Section 6.

(1) & (2). Determine the sign of the following permutations AND write the inverse of the permutation.

(a) $6=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$, can be written as:

$\tau_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$, $\tau_1 6 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \tau_2 \Rightarrow \tau_1 6 = \tau_2$ operating by $\tau_1^{-1}=\tau_1$

$\boxed{6 = \tau_1 \tau_2}$

We can check that $6 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \tau_1\tau_2$. Hence, $\epsilon(\theta) = (-1)^2 = 1$ even permutation.

the inverse is $6^{-1} = (\tau_1\tau_2)^{-1} = \tau_2^{-1}\tau_1^{-1} = \tau_2\tau_1 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

We can check that $66^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = 6^{-1}6$

(b) $6=\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, can be written as

$\tau_3 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, $\tau_3 6 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \tau_2 \Rightarrow \tau_3 6 = \tau_2$ operating by $\tau_3^{-1} = \tau_3$

$\boxed{6 = \tau_3\tau_2}$

We can check that $6 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \tau_3\tau_2$. Hence, $\epsilon(\theta) = (-1)^2 = 1$. even permutation.

the inverse is $6^{-1} = (\tau_3\tau_2)^{-1} = \tau_2^{-1}\tau_3^{-1} = \tau_2\tau_3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

We can check that $66^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = 6^{-1}6$

(c) $6 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = \tau_1 \Rightarrow \epsilon(6) = -1$. odd permutation (a transposition, in fact).

the inverse is $\tau_1^{-1} = \tau_1$. Check: $\tau_1\tau_1^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \tau_1^{-1}\tau_1$

(d) $\sigma=\begin{bmatrix}1&2&3&4\\2&3&1&4\end{bmatrix}$, can be written as:

$\tau_1=\begin{bmatrix}1&2&3&4\\2&1&3&4\end{bmatrix}$, $\tau_1\sigma=\begin{bmatrix}1&2&3&4\\1&3&2&4\end{bmatrix}=\tau_2 \Rightarrow \tau_1\sigma=\tau_2 \Rightarrow \sigma=\tau_1\tau_2$.

Check that: $\sigma=\begin{bmatrix}1&2&3&4\\2&3&1&4\end{bmatrix}=\tau_1\tau_2$. Hence $\epsilon(\sigma)=(-1)^2=1$. even permutation

the inverse is $\sigma^{-1}=(\tau_1\tau_2)^{-1}=\tau_2^{-1}\tau_1^{-1}=\tau_2\tau_1=\begin{bmatrix}1&2&3&4\\3&1&2&4\end{bmatrix}=\sigma^{-1}$

Check $\sigma\sigma^{-1}=\begin{bmatrix}1&2&3&4\\1&2&3&4\end{bmatrix}=\sigma^{-1}\sigma$

(e) $\sigma=\begin{bmatrix}1&2&3&4\\2&1&4&3\end{bmatrix}$, can be written as:

$\tau_1=\begin{bmatrix}1&2&3&4\\2&1&3&4\end{bmatrix}$, $\tau_1\sigma=\begin{bmatrix}1&2&3&4\\1&2&4&3\end{bmatrix}=\tau_2 \Rightarrow \tau_1\sigma=\tau_2 \Rightarrow \sigma=\tau_1\tau_2$.

Check: $\tau_1\tau_2=\begin{bmatrix}1&2&3&4\\2&1&4&3\end{bmatrix}=\sigma$. Hence $\epsilon(\sigma)=(-1)^2=1$. even permutation

the inverse is: $\sigma^{-1}=(\tau_1\tau_2)^{-1}=\tau_2^{-1}\tau_1^{-1}=\tau_2\tau_1=\begin{bmatrix}1&2&3&4\\2&1&4&3\end{bmatrix}=\sigma^{-1}$

Check: $\sigma\sigma^{-1}=\begin{bmatrix}1&2&3&4\\1&2&3&4\end{bmatrix}=\sigma^{-1}\sigma$

(f) $\sigma=\begin{bmatrix}1&2&3&4\\3&2&4&1\end{bmatrix}$, can be written as

$\tau_3=\begin{bmatrix}1&2&3&4\\1&2&4&3\end{bmatrix}$, $\tau_3\sigma=\begin{bmatrix}1&2&3&4\\4&2&3&1\end{bmatrix}=\tau_4 \Rightarrow \tau_3\sigma=\tau_4 \Rightarrow \sigma=\tau_3\tau_4$

check $\tau_3\tau_4=\begin{bmatrix}1&2&3&4\\3&2&4&1\end{bmatrix}$ Hence $\epsilon(\sigma)=(-1)^2=1$. even permutation.

the inverse is $\sigma^{-1}=(\tau_3\tau_4)^{-1}=\tau_4^{-1}\tau_3^{-1}=\tau_4\tau_3=\begin{bmatrix}1&2&3&4\\4&2&1&3\end{bmatrix}=\sigma^{-1}$

check $\sigma\sigma^{-1}=\begin{bmatrix}1&2&3&4\\1&2&3&4\end{bmatrix}=\sigma^{-1}\sigma$

(g) $\sigma=\begin{bmatrix}1&2&3&4\\4&2&1&3\end{bmatrix}$, can be written as:

$\tau_1=\begin{bmatrix}1&2&3&4\\1&2&4&3\end{bmatrix}$, $\tau_1\sigma=\begin{bmatrix}1&2&3&4\\3&2&1&4\end{bmatrix}=\tau_2 \Rightarrow \tau_1\sigma=\tau_2 \Rightarrow \sigma=\tau_1\tau_2$.

check $\tau_1\tau_2=\begin{bmatrix}1&2&3&4\\4&2&1&3\end{bmatrix}=\sigma$. Hence, $\epsilon(\sigma)=(-1)^2=1$. even permutation.

The inverse is $\sigma^{-1}=(\tau_1\tau_2)^{-1}=\tau_2^{-1}\tau_1^{-1}=\tau_2\tau_1=\begin{bmatrix}1&2&3&4\\3&2&4&1\end{bmatrix}$. Check: $\sigma\sigma^{-1}=\begin{bmatrix}1&2&3&4\\1&2&3&4\end{bmatrix}=\sigma^{-1}\sigma$.

(h) $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{bmatrix}$, can be written as:

$\tau_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{bmatrix}$, $\tau_1\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{bmatrix}$, $\tau_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{bmatrix}$, $\tau_2\tau_1\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{bmatrix} = \tau_3$

$\Rightarrow \tau_2\tau_1\sigma = \tau_3 \Rightarrow \sigma = \tau_1\tau_2\tau_3$. Check:

$\tau_1\tau_2\tau_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{bmatrix} = \sigma$. Hence, $\epsilon(\sigma) = (-1)^3 = -1$. odd permutation.

the inverse is $\sigma^{-1} = (\tau_1\tau_2\tau_3)^{-1} = \tau_3\tau_2\tau_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{bmatrix}$. check:

$\sigma\sigma^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} = \sigma^{-1}\sigma$

(i) $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{bmatrix}$, can be written as:

$\tau_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{bmatrix}$, $\tau_1\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{bmatrix}$, $\tau_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{bmatrix}$, $\tau_2\tau_1\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{bmatrix} = \tau_3$

$\Rightarrow \tau_2\tau_1\sigma = \tau_3 \Rightarrow \sigma = \tau_1\tau_2\tau_3$. Check

$\tau_1\tau_2\tau_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{bmatrix} = \sigma$. Hence, $\epsilon(\sigma) = (-1)^3 = -1$. Odd permutation

the inverse is $\sigma^{-1} = (\tau_1\tau_2\tau_3)^{-1} = \tau_3\tau_2\tau_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{bmatrix}$

$\sigma\sigma^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} = \sigma^{-1}\sigma$.

---

3. Show that the number of odd permutations of $\{1, \ldots, n\}$ for $n \geq 2$ is equal to the number of even permutations.

claim: Let τ be a transposition. Show that the map $\sigma \mapsto \tau\sigma$ establishes an injective and surjective (A.K.A. bijective) map between the even an odd permutations. the bijection means that the set of odd permutations has the same number of elements than the set of even permutations.

Pf: Let $ODD = \{\sigma \mid \sigma$ is an odd permutation$\}$ $EVEN = \{\sigma' \mid \sigma'$ is an even permut$\}$

Let $f: EVEN \to odd$, by $f(\sigma) = \tau\sigma$, where τ is a transposition.

NOTE that any element of ODD OR EVEN can be written as the composition of finitely many transpositions.

f is injective: Suppose that $f(\sigma_1) = f(\sigma_2)$, for $\sigma_1, \sigma_2 \in$ EVEN.

$\Rightarrow \tau\sigma_1 = \tau\sigma_2$. Operating by the inverse of τ, i.e., τ^{-1} on both sides we obtain $\sigma_1 = \sigma_2$. Hence, f is injective.

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f is surjective: Given $\sigma \in ODD$, we can write it as:

$$\sigma = \tau_1 \tau_2 \cdots \tau_s, \quad s \text{ is an ODD number.}$$

If we operate by a transposition τ on both sides:

$$\tau\sigma = \tau\tau_1\tau_2\cdots\tau_s = f(\sigma') \Rightarrow \sigma' \in EVEN.$$

Hence, for any $\sigma \in ODD$, there exists $\sigma' \in EVEN$ such that $f(\sigma') = \sigma \Rightarrow$ f is surjective & injective $\Leftrightarrow$ f is bijective. $\Rightarrow \#EVEN = \#ODD$.

Section 6.7

(1) Show that when $n=3$, the expansion of theorem 7.2. is the six-term expression given in §2, i.e., show that

Pf: $D(A^1, A^2, A^3) = Det(A)$.

$D(A^1, A^2, A^3) = \sum_{\sigma} \epsilon(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} a_{\sigma(3),3} = \ldots$ By theorem 7.2.

where $\sigma = \{f \mid f \in Aut(\{1,2,3\})\}$. there are $3! = 3\cdot2\cdot1 = 6$ different fs. (i) $f(1)=1, f(2)=2, f(3)=3$, (ii) $f(1)=2, f(2)=1, f(3)=3$ (iii) $f(1)=3, f(2)=2, f(3)=1$, (iv) $f(1)=1, f(2)=3, f(3)=2$ (v) $f(1)=2, f(2)=3, f(3)=1$ (vi) $f(1)=3, f(2)=1, f(3)=2$

the sign of each of these are: (i) $+1$, (ii) -1, (iii) -1, (iv) -1, (v) $+1$, (vi) $+1$. (the same # of odd as even permutation as proved in a previous exercise). Hence

$$D(A^1, A^2, A^3) = (+1)a_{11}a_{22}a_{33} + (-1)a_{21}a_{12}a_{33} + (-1)a_{31}a_{22}a_{13} + (-1)a_{11}a_{32}a_{23} + (+1)a_{21}a_{32}a_{13} + (+1)a_{31}a_{12}a_{23}$$

$$= a_{11}a_{22}a_{33} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} = Det(A)$$ as defined in the six-term expression given in §2.

Section 6.7

(3) (a) $F(A^1, \ldots, A^n) = det(C) F(B^1, \ldots, B^n)$, where $C = (c_{ij})$.

$\Rightarrow F(A^1, \ldots, A^n) = 5 \cdot (-3) = -15$.

(b) $F(A^1, \ldots, A^n) = det(a_{ij}) F(E^1, \ldots, E^n)$ But $det(a_{ij}) = D(A^1, \ldots, A^n)$

$\Rightarrow D(A^1, \ldots, A^n) = 5$