Enrique Areyan M409 - Homework 5

Section 6.5

(1)(a) J_{rs} is a $n\times n$ matrix whose rs-component is 1, where $1\le r, s\le n$ and $r\neq s$, and all other components are 0. In other worsds, J_{rs} is a matrix with a one in a non-diagonal entry and 0 every where else. If $E_{rs}=I+J_{rs}$, then E_{rs} is a matrix with a one in its diagonal and in another, non-diagonal entry. From theory we know that column operations do not change $|D(E_{rs})|$. Take the $(-1)r$ col of E_{rs} and add it to the s col of E_{rs}. You obtain I back. But, by previous work $DET(I)=1=D(E_{rs})$. we did not swap two columns, so the sign remains the same.

(1)(b) A $n\times n$ matrix. the effect of multipying $E_{rs}\cdot A$ is to obtain the matrix A such that the the row s of A is added to the row r. Similarly, the product AE_{rs} results in matrix A such that the col r of A is added to the col s.

(2) If A is a triangular matrix, (square), then its columns look like:

$$A^1=\begin{bmatrix} w_{11}\\ 0\\ \vdots\\ 0\end{bmatrix}, A^2=\begin{bmatrix} w_{12}\\ w_{22}\\ 0\\ \vdots\\ 0\end{bmatrix},\dots, A^n=\begin{bmatrix} w_{1n}\\ w_{2n}\\ \vdots\\ w_{nn}\end{bmatrix}.$$

this is upper triangular, but the reasoning is pretty much the same for lower triangular also.

We want to prove that the A^i's are L.I iff $w_{ii}\neq 0$ $\forall i: 1\le i\le n$.

($\Leftarrow$) we can reduce this problem to solving:

$$\left.\begin{aligned} c_1w_{11}&=0\\ c_1w_{12}+c_2w_{22}&=0\\ &\vdots\\ c_1w_{1n}+c_2w_{2n}+\dots+c_nw_{nn}&=0\end{aligned}\right\}(*)\Rightarrow$$

By hypothesis $w_{ii}\neq 0$, hence: $c_i=0$ $\forall i$ $\Rightarrow$ A^i are L.I. we simply substitute equation by equation.

($\Rightarrow$) Suppose $\exists i$ such that $w_{ii}=0$. and that A^i's are L.I. then we can see from (*) that A will have a non trivial solution Hence $\dim(\ker(A))>0$. But $n=\dim(\ker(A))+\dim(\mathrm{Img}(A))$ and A^i's are L.I and there are n of them. thus, $\{A^1,\dots,A^n\}$ form a basis. $\Rightarrow \dim(\mathrm{Img}(A))=n$, which contradicts the fact that $\dim(\ker(A))>0$. therefore, it must be the case that $w_{ii}\neq 0$ $\forall i$.

Additional Exercises

(2) (a) Let $B: \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ be a bilinear form, such that $B(v,v)=0$ $\forall v \in V$. Show that $B(v,w) = -B(w,v)$ $\forall v,w \in V$.

By hypothesis $\quad 0 = B(v+w, v+w)$

By linearity $\quad B(v+w, v+w) = B(v,v) + B(v,w) + B(w,v) + B(w,w)$

By hypothesis both $B(v,v)=0$ and $B(w,w)=0$. Hence,

$$0 = B(v,w) + B(w,v) \Rightarrow B(v,w) = -B(w,v).$$

(b) Let $F: (\mathbb{K}^n)^n \to \mathbb{K}$ be a multilinear form so that if $v_i = v_j$ with $i \neq j$, then $F(v_1, \dots, v_n) = 0$. Show that F is alternating.

Solution: F is alternating means $F(v_1, \dots, v_j, v_{j+1}, \dots, v_n) = -F(v_1, \dots, v_{j+1}, v_j, \dots, v_n)$.

$F(v_1, \dots, v_j + v_{j+1}, v_{j+1}, \dots, v_n) =$ By linearity.

$F(v_1, \dots, v_j, v_{j+1}, \dots, v_n) + F(v_1, \dots, v_{j+1}, v_{j+1}, \dots, v_n)$

But, by hypothesis, $F(v_1, \dots, v_{j+1}, v_{j+1}, \dots, v_n) = 0$.

Also,

$F(v_1, \dots, v_j, v_j + v_{j+1}, \dots, v_n) =$ By linearity

$F(v_1, \dots, v_j, v_j, \dots, v_n) + F(v_1, \dots, v_j, v_{j+1}, \dots, v_n)$

But, by hypothesis, $F(v_1, \dots, v_j, v_j, \dots, v_n) = 0$

Hence, $F(v_1, \dots, v_j, v_{j+1}, \dots, v_n) = -F(v_1, \dots, v_{j+1}, v_j, \dots, v_n)$

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the addionat exercise (corrected) on permutation is on the last page.

Section 8.1

(1) Let $a \in \mathbb{K}$ and $a \neq 0$. Prove that the eigenvectors of the matrix

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = A$$

generate a 1-dimensional space, and give a basis for this space

Solution: An eigenvector v is such that: $Av = \lambda v$, where $\lambda \in \mathbb{K}$.

$$\Rightarrow \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix} \Rightarrow \begin{array}{l} v_1 + a v_2 = \lambda v_1 \\ v_2 = \lambda v_2 \Rightarrow \lambda = 1. \text{ or } v_2 = 0 \end{array}$$

If $v_2 = 0 \Rightarrow v_1 = \lambda v_1 \Rightarrow \lambda = 1$. Hence, $\lambda = 1$ always.

$v_1 + a v_2 = v_1 \Rightarrow a v_2 = 0$, But $a \neq 0$, hence $v_2 = 0$

the eigen vectors belong to the space genereted by $\langle \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \} \rangle$.

(3) Let A be a diagonal matrix with diagonal elements $a_{11}, \dots, a_{nn}$. What is the dimension of the space generated by the eigenvectors of A? Exhibit a basis for the space, and give the eigen values.

Solution:

$$Av = \lambda v \Rightarrow \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & & a_{nn} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} a_{11} v_1 = \lambda v_1 \\ a_{22} v_2 = \lambda v_2 \\ \vdots \\ a_{nn} v_n = \lambda v_n \end{array} \qquad \begin{array}{ll} w_1 = e_1 & \lambda_1 = a_{11} \\ w_2 = e_2 & \lambda_2 = a_{22} \\ \vdots & \vdots \\ w_n = e_n & \lambda_n = a_{nn} \end{array}$$

the dimension is n.
the basis is $\{e_1, \dots, e_n\}$.
the eigen values are the diagonal elements,

i.e., each e_i have eigen value a_{ii}.

(4) Let $A = (a_{ij})$ be an $n \times n$ matrix such that for each $i = 1, \dots, n$ we have $\sum_{j=1}^{n} a_{ij} = 0$. Show that 0 is an eigen value of A.

Solution: A is such that the sum of the element in each row is zero

0 is an eigen value of A iff, $Av = 0v$, i.e.,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nn}v_n \end{bmatrix}, \text{ if we take } \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

then

$$A\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} + a_{12} + \cdots + a_{1n} \\ a_{21} + a_{22} + \cdots + a_{2n} \\ \vdots \\ a_{n1} + a_{n2} + \cdots + a_{nn} \end{bmatrix} = \text{By hypothesis} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Hence, $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is eigenvector with eigen value 0.

(5) (a) Show that if $\theta \in \mathbb{R}$, then the matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

always has an eigen vector in $\mathbb{R}^2$, and in fact that there exists a vector v_1 such that $Av_1 = v_1$.

Solution: $Av = \lambda v \Rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Rightarrow \left.\begin{array}{l} \cos\theta\, x_1 + \sin\theta\, x_2 = \lambda x_1 \\ \sin\theta\, x_1 - \cos\theta\, x_2 = \lambda x_2 \end{array}\right\} (*)$$

$\Rightarrow$ where $\cos\theta \neq 1$.

Using the hint, let $x_1 = \dfrac{\sin\theta}{1-\cos\theta}$ and $x_2 = x_2$. Replacing into $(*)$:

let $\lambda = 1$: $\begin{cases} \cos\theta\left(\dfrac{\sin\theta}{1-\cos\theta}\right) + \sin\theta\, x_2 = \dfrac{\sin\theta}{1-\cos\theta} & (1) \\ \sin\theta\left(\dfrac{\sin\theta}{1-\cos\theta}\right) - \cos\theta\, x_2 = x_2 & (2) \end{cases}$

From (2): $\dfrac{\sin^2\theta}{1-\cos\theta} = x_2 + \cos\theta\, x_2 = x_2(1+\cos\theta)$

$$\Rightarrow \frac{\sin^2\theta}{(1-\cos\theta)(1+\cos\theta)} = x_2 = \frac{\sin^2\theta}{1+\cancel{\cos\theta}-\cancel{\cos\theta}-\cos^2\theta} = \frac{\sin^2\theta}{1-\cos^2\theta} = \frac{\sin^2\theta}{\sin^2\theta} = 1$$

Hence, $\boxed{x_2 = 1}$. One can verify that $x = \begin{pmatrix} \frac{\sin\theta}{1-\cos\theta} \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue 1, i.e; $Ax = x$ (if $\cos\theta \neq 1$).

$$Ax = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \frac{\sin\theta}{1-\cos\theta} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\cos\theta\sin\theta}{1-\cos\theta} + \sin\theta \\ \frac{\sin^2\theta}{1-\cos\theta} - \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{\cancel{\cos\theta\sin\theta} + \sin\theta - \cancel{\cos\theta\sin\theta}}{1-\cos\theta} \\ \frac{\sin^2\theta - \cos\theta + \cos^2\theta}{1-\cos\theta} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sin\theta}{1-\cos\theta} \\ \frac{1-\cos\theta}{1-\cos\theta} \end{pmatrix} = \begin{pmatrix} \frac{\sin\theta}{1-\cos\theta} \\ 1 \end{pmatrix} = x.$$

If $\cos\theta = 1$, then $\cos^2\theta = 1 \Rightarrow \sin^2\theta = 0 \Rightarrow \sin\theta = 0$. the matrix A is now: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the eigen vector would be:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{array}{l} x_1 = \lambda x_1 \Rightarrow \lambda = 1 \\ -x_2 = \lambda x_2 \end{array} \Rightarrow x_2 = 0$$

the eigen vectors are $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$, with eigenvalue 1.

(5) (b) Let $v_2 \in \mathbb{R}^2$ perpendicular to $v_1 = \begin{pmatrix} \frac{\sin\theta}{1-\cos\theta} \\ 1 \end{pmatrix}$. Show that $Av_2 = -v_2$. Define this to mean that A is a reflection.

Solution: $v_1 \cdot v_2 = 0 \iff \left(\frac{\sin\theta}{1-\cos\theta}, 1\right) \cdot (v_{21}, v_{22}) = 0$.

$\iff \frac{\sin\theta\, v_{21}}{1-\cos\theta} + v_{22} = 0$

If $\cos\theta \neq 1$, then $(v_{21}, v_{22}) = (1-\cos\theta, -\sin\theta)$ is a vector perpendicular to v_1, i.e., $\frac{\sin\theta}{\cancel{1-\cos\theta}}(\cancel{1-\cos\theta}) - \sin\theta = \sin\theta - \sin\theta = 0$.

TAKE $A.v_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} 1-\cos\theta \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta(1-\cos\theta) - \sin^2\theta \\ \sin\theta(1-\cos\theta) + \cos\theta\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta - (\cos^2\theta + \sin^2\theta) \\ \sin\theta - \sin\theta\cos\theta + \sin\theta\cos\theta \end{pmatrix}$

$= \begin{pmatrix} -1 + \cos\theta \\ \sin\theta \end{pmatrix} = -1\begin{pmatrix} 1-\cos\theta \\ -\sin\theta \end{pmatrix} = -v_2$.

If $\cos\theta = 1$, then $(v_{21}, v_{22}) = (0,1)$, hence,

$Av_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -v_2$

A reflects v in the line perpendicular to it.

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(6) Let $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ be the matrix of rotation.

Show that $R(\theta)$ does not have any real eigenvalues.

Solution: To find the real eigenvalues λ we need to solve:

$$\det(R(\theta) - \lambda I) = 0 \iff \det\left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$$

$\iff \det\begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} = 0 \iff (\cos\theta - \lambda)^2 + \sin^2\theta = 0 \iff$

$\cos^2\theta - 2\cos\theta\lambda + \lambda^2 + \sin^2\theta = 0 \iff \lambda^2 - 2\cos\theta\lambda + 1 = 0$

We attempt to solve this quadratic equation:

$$\frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \Rightarrow \text{the discriminat} = 4(\cos^2\theta - 1)$$

Hence, If $\cos^2\theta < 1 \Rightarrow 4(\cos^2\theta - 1) < 0 \Rightarrow$ negative discriminant, no real solution.

If $\cos^2\theta = 1 \Rightarrow 4(1-1) = 0 \Rightarrow$ there is a real solution if $\cos^2\theta = 1$

But, if $\cos^2\theta = 1$ than $\sin\theta^2 = 0$ $\Rightarrow \cos\theta = \pm 1$ and $\sin\theta = 0$

$\Rightarrow R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, i.e, $R = \pm I$.

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(7) Let V be a f.d. v.s. Let $A: V \to V$ and $B: V \to V$ be both linear. Assume $AB = BA$. Show that if v is an eigenvector of A, with eigenvalue λ, then Bv is an eigenvector of A, with eigenvalue λ also if $Bv \neq 0$.

<u>Solution</u>: By definition, v is an eigenvector of A with eigenvalue λ iff $Av = \lambda v$. Here we can apply B to both sides:

$B(Av) = B(\lambda v)$. B is a linear map, thus the scalar λ comes out:

$B(Av) = \lambda(Bv)$. By composition of linear maps

$(BA)v = \lambda(Bv)$. By hypothesis $BA = AB$

$(AB)v = \lambda(Bv)$. By composition of linear maps

$A(Bv) = \lambda(Bv)$. By definition of eigenvector, Bv is an eigenvector of A with eigenvalue λ. $(Bv \neq 0)$

<u>Section 8.2</u>:

(1) Let A be a diagonal matrix.

(a) what is the characteristic polynomial of A?

$P_A(t) = Det(A - tI) = Det\begin{pmatrix} a_1 - t & 0 & \cdots & 0 \\ 0 & a_2 - t & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & a_n - t \end{pmatrix}$ by previous work we know that the determinant of a diagonal matrix is the product of its diagonal entries, hence $\boxed{P_A(t) = (a_1 - t)(a_2 - t)\cdots(a_n - t)}$

(b) what are its eigen values?

An eigenvalue is such that $P_A(t) = 0 \Rightarrow (a_1 - t)(a_2 - t)\cdots(a_n - t) = 0$ the eigenvalues are $a_1, \ldots, a_n$, the diagonal entries.

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(2) Let A be a lower-triangular matrix. What is the characteristic polinomial of A, and what are its eigenvalues?

Same as before: $P_A(t) = Det(A - tI) = Det\begin{pmatrix} a_{11} - t & 0 & \cdots & 0 \\ a_{21} & a_{22} - t & \cdots & 0 \\ & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ by previous work:

$P_A(t) = (a_{11} - t)(a_{22} - t)\cdots(a_{nn} - t)$. Eigen values are $a_{11}, a_{22}, \ldots, a_{nn}$.

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(5) Find the eigenvalues and eigenvectors of the following matrices. Show that the eigen vectors form a 1-dimensional space.

(a) $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} = A$. $P_A(t) = \det(A - tI) = \det\left(\begin{pmatrix} 2-t & -1 \\ 1 & -t \end{pmatrix}\right) = (2-t)(-t)+1 = t^2 - 2t + 1$.

the eigenvalue t is such that $P_A(t) = 0 \iff t^2 - 2t + 1 = 0$

$\iff (t-1)^2 = 0 \implies \boxed{t = 1}$ with multiplicity two.

<u>eigenvalue +1</u>:

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \implies \begin{cases} 2x_1 - x_2 = x_1 & \implies 2x_2 - x_2 = x_1 \implies x_2 = x_1. \\ x_1 = x_2 \end{cases}$$

$V_{+1} = \left\{ \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} \mid x_1 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$, a one-dimensional space.

(b) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A \implies P_A(t) = \det(A - tI) = \det\begin{pmatrix} 1-t & 1 \\ 0 & 1-t \end{pmatrix} = (1-t)^2$

the eigenvalue t is such that $P_A(t) = 0 \iff (1-t)^2 = 0 \implies t = 1$.

<u>eigenvalue +1</u>:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \implies \begin{cases} x_1 + x_2 = x_1 \implies x_2 = 0 \\ x_2 = x_2 \end{cases}$$

$V_{+1} = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \mid x_1 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$, a one-dimensional space

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(c) $\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} = A \implies P_A(t) = \det(A - tI) = \det\begin{pmatrix} 2-t & 0 \\ 1 & 2-t \end{pmatrix} = (2-t)^2$

the eigenvalue t is such that $P_A(t) = 0 \iff (2-t)^2 = 0 \implies t = 2$.

<u>eigenvalue +2</u>:

$$\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \implies \begin{cases} 2x_1 = 2x_1 \\ x_1 + 2x_2 = 2x_2 \implies x_1 = 0 \end{cases}$$

$V_{+2} = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \mid x_2 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$, a one-dimensional space

we can check this:

$$A \cdot v = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2x_2 \end{pmatrix} = 2\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = 2v.$$

(d) $\begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix} = A \Rightarrow P_A(t) = \det(A - tI) = \det\begin{pmatrix} 2-t & -3 \\ 1 & -1-t \end{pmatrix}$

$= (2-t)(-1-t)+3 = -2-2t+t+t^2+3 = t^2-t+1$

to find eigenvalues, set $P_A(t)=0 \Leftrightarrow t^2-t+1=0$

Solving the quadratic equation:

$$\frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}\, i}{2}$$

We have two eigen values: $\underbrace{\frac{1+\sqrt{3}i}{2}}^{=\lambda_1}$ and $\underbrace{\frac{1-\sqrt{3}i}{2}}^{=\lambda_2}$

<u>eigenvalue λ:</u>

$$\begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} 2x_1 - 3x_2 = \lambda_1 x_1 \\ x_1 - x_2 = \lambda_1 x_2 \end{cases}$$

$$\Rightarrow \begin{cases} 2x_1 - \lambda_1 x_1 = 3x_2 \Rightarrow x_1(2-\lambda) = 3x_2 \\ x_1 = \lambda_1 x_2 + x_2 \Rightarrow x_1 = x_2(1+\lambda) \end{cases}$$

Let $x_2 = 1$, then $\Rightarrow$ $x_1 = (1+\lambda)$, Hence

$V_{\lambda_1} = \left\{ \begin{pmatrix} 1+\lambda_1 \\ x_2 \end{pmatrix} \middle| x_2 \in \mathbb{C} \right\} = \left\langle \begin{pmatrix} \frac{3+\sqrt{3}i}{2} \\ 1 \end{pmatrix} \right\rangle$ Both are one-dimensional v.s.

$V_{\lambda_2} = \left\{ \begin{pmatrix} 1+\lambda_2 \\ x_2 \end{pmatrix} \middle| x_2 \in \mathbb{C} \right\} = \left\langle \begin{pmatrix} \frac{3-\sqrt{3}i}{2} \\ 1 \end{pmatrix} \right\rangle$

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(6) (a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $P_A(t) = \det\begin{pmatrix} 1-t & 1 & 1 \\ 0 & 1-t & 1 \\ 0 & 0 & 1-t \end{pmatrix}$

$= (1-t)\det\begin{pmatrix} 1-t & 1 \\ 0 & 1-t \end{pmatrix} - 1\det\begin{pmatrix} 0 & 1 \\ 0 & 1-t \end{pmatrix} + 1\det\begin{pmatrix} 0 & 1-t \\ 0 & 0 \end{pmatrix} = (1-t)^3$

$\Rightarrow$ the only eigen value is $t=1$.

<u>eigenvectors for eigen value $t=1$:</u>

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{matrix} x_1+x_2+x_3 = x_1 \Rightarrow x_1+x_2 = x_1 \Rightarrow x_2 = 0 \\ x_2+x_3 = x_2 \Rightarrow x_3 = 0 \\ x_3 = x_3 \end{matrix}$$

$V_1 = \left\{ \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} \middle| x_1 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$. We can check:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$$

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8.2.6 (b) $A=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix} \Rightarrow P_A(t)=\det(A-tI)=\det\begin{pmatrix}1-t&1&0\\0&1-t&1\\0&0&1-t\end{pmatrix}$

$=(1-t)\det\begin{pmatrix}1-t&1\\0&1-t\end{pmatrix}-1\det\begin{pmatrix}0&1\\0&1-t\end{pmatrix}+0.\det(\;)=(1-t)^3$.

the only eigenvalue is $t=1$.

$\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \Rightarrow \begin{matrix}x_1+x_2=x_1 \Rightarrow x_2=0\\ x_2+x_3=x_2 \Rightarrow x_3=0\\ x_3=x_3\end{matrix}$

$V_1=\left\{\begin{pmatrix}x_1\\0\\0\end{pmatrix} \mid x_1\in\mathbb{R}\right\}=\left\langle\begin{pmatrix}1\\0\\0\end{pmatrix}\right\rangle$, a one dimensional vector space.

8.2.6 (7) (a)

$A=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix} \Rightarrow P_A(t)=\det(A-tI)=\det\begin{pmatrix}-t&1&0&0\\0&-t&1&0\\0&0&-t&1\\1&0&0&-t\end{pmatrix}$

$=-t\det\begin{pmatrix}-t&1&0\\0&-t&1\\0&0&-t\end{pmatrix}-1\det\begin{pmatrix}0&1&0\\0&-t&1\\1&0&-t\end{pmatrix}$

$=(-t)(-t)t^2-1(-1)\det\begin{pmatrix}0&1\\1&-t\end{pmatrix}=\boxed{t^4-1}$

To find eigenvalues, set $P_A(t)=0 \Leftrightarrow t^4-1=0 \Rightarrow \boxed{t=\pm 1}$

eigenvalue $+1$:

$\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \Rightarrow \begin{cases}x_2=x_1\\x_3=x_2 \Rightarrow x_1=x_3=x_4=x_2\\x_4=x_3\\x_1=x_4\end{cases}$

$V_{+1}=\left\{\begin{pmatrix}x_1\\x_1\\x_1\\x_1\end{pmatrix} \mid x_1\in\mathbb{R}\right\}=\left\langle\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\rangle$, we can check:

$\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_1\\x_1\\x_1\end{pmatrix}=\begin{pmatrix}x_1\\x_1\\x_1\\x_1\end{pmatrix}=1\cdot\begin{pmatrix}x_1\\x_1\\x_1\\x_1\end{pmatrix}$.

eigenvalue -1:

$\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}-x_1\\-x_2\\-x_3\\-x_4\end{pmatrix} \Rightarrow \begin{cases}x_2=-x_1 \Rightarrow x_1=-x_2=-x_4=x_3\\x_3=-x_2 \Rightarrow x_3=-x_4=-x_2\\x_4=-x_3\\x_1=-x_4\end{cases}$

$V_{-1}=\left\{\begin{pmatrix}x_1\\-x_1\\x_1\\-x_1\end{pmatrix}, x_1\in\mathbb{R}\right\}=\left\langle\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\right\rangle$

8.2.7.

(b) $A=\begin{pmatrix}-1&0&1\\-1&3&0\\-4&13&-1\end{pmatrix} \Rightarrow P_A(t)=\det(A-tI)=\det\begin{pmatrix}-1-t&0&1\\-1&3-t&0\\-4&13&-1-t\end{pmatrix}$

$=(-1-t)\det\begin{pmatrix}3-t&0\\13&-1-t\end{pmatrix}+1\cdot\det\begin{pmatrix}-1&3-t\\-4&13\end{pmatrix}$

$=(-1-t)(3-t)(-1-t)+(-13)-((-4)(3-t))$

$=(-1-t)^2(3-t)-13+12-4t=(1+2t+t^2)(3-t)-1-4t$

$=\cancel{3}+\cancel{6t}+3t^2-\cancel{t}-\cancel{2t^2}-t^3-\cancel{1}-4t=-t^3+t^2+t+2$

Eigenvalues: $P_A(t)=0 \Leftrightarrow -t^3+t^2+t+2=0 \Leftrightarrow t^3-t^2-t-2=0$

this polinomial has a unique real root $t=2$.

Eigenvectors:

$\begin{pmatrix}-1&0&1\\-1&3&0\\-4&13&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}2x_1\\2x_2\\2x_3\end{pmatrix} \Rightarrow \begin{cases}-x_1+x_3=2x_1 \Rightarrow x_3=3x_1 \Rightarrow x_3=3x_2\\-x_1+3x_2=2x_2 \Rightarrow -x_1=-x_2 \Rightarrow x_1=x_2\\-4x_1+13x_2-x_3=2x_3\end{cases}$

$V_2=\left\{\begin{pmatrix}x_1\\x_1\\3x_1\end{pmatrix} \mid x_1\in\mathbb{R}^2\right\}=\left\langle\begin{pmatrix}1\\1\\3\end{pmatrix}\right\rangle$. we can check this:

$\begin{pmatrix}-1&0&1\\-1&3&0\\-4&13&-1\end{pmatrix}\begin{pmatrix}x_1\\x_1\\3x_1\end{pmatrix}=\begin{pmatrix}-x_1+3x_1\\-x_1+3x_1\\-4x_1+13x_1-3x_1\end{pmatrix}=\begin{pmatrix}2x_1\\2x_1\\6x_1\end{pmatrix}=2\begin{pmatrix}x_1\\x_1\\3x_1\end{pmatrix}$

8.2.8

(a) $A=\begin{pmatrix}2&4\\5&3\end{pmatrix} \Rightarrow P_A(t)=\det(A-tI)=\det\begin{pmatrix}2-t&4\\5&3-t\end{pmatrix}=(2-t)(3-t)-20$

$=6-2t-3t+t^2-20=t^2-5t-14$.

Eigenvalues: $P_A(t)=0 \Leftrightarrow t^2-5t-14=0 \Rightarrow \frac{5\pm\sqrt{25+56}}{2}=\frac{5\pm\sqrt{81}}{2}$

$\Rightarrow \frac{5\pm9}{2} \Rightarrow t_1=7,\ t_2=-2$.

V_7: $\begin{pmatrix}2&4\\5&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=7\begin{pmatrix}x_1\\x_2\end{pmatrix} \Rightarrow \begin{cases}2x_1+4x_2=7x_1 \Rightarrow 4x_2=5x_1 \Rightarrow x_1=\frac{4}{5}x_2\\5x_1+3x_2=7x_2 \Rightarrow 5x_1=4x_2 \Rightarrow x_2=\frac{5}{4}x_1\end{cases}$

$V_7=\left\{\begin{pmatrix}x_1\\5/4\,x_1\end{pmatrix} \mid x_1\in\mathbb{R}\right\}=\left\langle\begin{pmatrix}1\\5/4\end{pmatrix}\right\rangle$. we can check:

$\begin{pmatrix}2&4\\5&3\end{pmatrix}\begin{pmatrix}x_1\\5/4\,x_1\end{pmatrix}=\begin{pmatrix}2x_1+5x_1\\5x_1+\frac{15}{4}x_1\end{pmatrix}=\begin{pmatrix}7x_1\\\frac{35}{4}x_1\end{pmatrix}=7\begin{pmatrix}x_1\\\frac{5}{4}x_1\end{pmatrix}$

$V_{-2}: \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} 2x_1+4x_2=-2x_1 \Rightarrow 4x_2=-4x_1 \Rightarrow x_2=-x \\ 5x_1+3x_2=-2x_2 \Rightarrow 5x_1=-5x_2 \end{cases}$

$V_{-2}=\left\{\begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} \mid x_1\in\mathbb{R}\right\} = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle$ we can check:

$\begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = \begin{pmatrix} 2x_1-4x_1 \\ 5x_1-3x_1 \end{pmatrix} = \begin{pmatrix} -2x_1 \\ 2x_1 \end{pmatrix} = -2\begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}$

(b) $A=\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$, $P_A(t)=\det(A-tI)=\det\begin{pmatrix} 1-t & 2 \\ 2 & -2-t \end{pmatrix} = (1-t)(-2-t)-4$

$= -2-t+2t+t^2-4 = t^2+t-6 = (t-2)(t+3)$

Eigenvalues: $P_A(t)=0 \iff (t-2)(t+3)=0 \Rightarrow \boxed{t=2}$ or $\boxed{t=-3}$

$V_2: \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \Rightarrow \begin{cases} x_1+2x_2=2x_1 \Rightarrow 2x_2=x_1 \Rightarrow x_2=\frac{1}{2}x_1 \\ 2x_1-2x_2=2x_2 \Rightarrow 2x_1=4x_2 \Rightarrow x_1=2x_2 \end{cases}$

$V_2=\left\{\begin{pmatrix} x_1 \\ \frac{1}{2}x_1 \end{pmatrix} \mid x_1\in\mathbb{R}\right\} = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$. we can check:

$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ \frac{1}{2}x_1 \end{pmatrix} = \begin{pmatrix} x_1+x_1 \\ 2x_1-x_1 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_1 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ \frac{1}{2}x_1 \end{pmatrix}$.

$V_{(-3)}: \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3x_1 \\ -3x_2 \end{pmatrix} \Rightarrow \begin{cases} x_1+2x_2=-3x_1 \Rightarrow 2x_2=-4x_1 \Rightarrow x_2=-2x_1 \\ 2x_1-2x_2=-3x_2 \Rightarrow 2x_1=-x_2 \Rightarrow x_2=-2x_1 \end{cases}$

$V_{(-3)}=\left\{\begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix} \mid x_1\in\mathbb{R}\right\} = \left\langle \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle$. we can check:

$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix} = \begin{pmatrix} x_1-4x_1 \\ 2x_1+4x_1 \end{pmatrix} = \begin{pmatrix} -3x_1 \\ 6x_1 \end{pmatrix} = -3\begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}=A \Rightarrow P_A(t)=\det(A-tI)=\det\begin{pmatrix} 3-t & 2 \\ -2 & 3-t \end{pmatrix}$

$= (3-t)^2+4 = 9-6t+t^2+4 = t^2-6t+13$.

Eigenvalues: $P_A(t)=0 \iff t^2-6t+13=0$. Using quadratic solver:

$\frac{6\pm\sqrt{36-52}}{2} = \frac{6\pm\sqrt{-16}}{2} = \frac{6\pm 4i}{2} \Rightarrow \begin{cases} 3+2i=t_1 \\ 3-2i=t_2 \end{cases}$

$V_{t_1}: \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} 3x_1+2x_2=t_1x_1 \Rightarrow 2x_2=(t_1-3)x_1 \\ -2x_1+3x_2=t_1x_2 \Rightarrow -2x_1=(t_1-3)x_2 \end{cases}$

If $x_1 = 1$ then $x_2 = \frac{t_1 - 3}{2}$, the same holds for t_2, hence

$$V_{t_1} = \left\{ \begin{pmatrix} x_1 \\ \frac{t_1-3}{2} \end{pmatrix} = \begin{pmatrix} x_1 \\ \frac{3+2i-3}{2} \end{pmatrix} = \begin{pmatrix} x_1 \\ i \end{pmatrix} \right\} = \left\langle \begin{pmatrix} 1 \\ i \end{pmatrix} \right\rangle.$$ we can check:

$$\begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ i \end{pmatrix} = \begin{pmatrix} 3x_1 + 2i \\ -2x_1 + 3i \end{pmatrix} = 3+2i \begin{pmatrix} x_1 \\ i \end{pmatrix}$$

$$V_{t_2} = \left\{ \begin{pmatrix} x_1 \\ \frac{t_1-3}{2} \end{pmatrix} = \begin{pmatrix} x_1 \\ \frac{3-2i-3}{2} \end{pmatrix} = \begin{pmatrix} x_1 \\ -i \end{pmatrix} \right\} = \left\langle \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\rangle.$$

(d) $\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix} = A \Rightarrow P_A(t) = \det(A - tI) = \det\begin{pmatrix} -1-t & 2 & 2 \\ 2 & 2-t & 2 \\ -3 & -6 & -6-t \end{pmatrix}$

$= (-1-t)\det\begin{pmatrix} 2-t & 2 \\ -6 & -6-t \end{pmatrix} - (2)\det\begin{pmatrix} 2 & 2 \\ -3 & -6-t \end{pmatrix} + (2)\det\begin{pmatrix} 2 & 2-t \\ -3 & -6 \end{pmatrix}$

$= (-1-t)[(2-t)(-6-t)+12] - 2[-12-2t+6] + 2[-12-(-6+3t)]$

$= (-1-t)[-12-2t+6t+t^2+12] - 2[-2t-6] + 2[-12+6-3t]$

$= (-1-t)[t^2+4t] + 4t + 12 + 2[-6-3t]$

$= -t^2 - 4t - t^3 - 4t^2 + 4t + 12 - 12 - 6t$

$= -t^3 - 5t^2 - 6t.$

Eigenvalues: $P_A(t) = 0 \Leftrightarrow -t^3 - 5t^2 - 6t = 0 \Leftrightarrow t^3 + 5t^2 + 6t = 0$

$\Leftrightarrow t(t^2+5t+6) = 0 \Rightarrow t = 0$ or $t^2 + 5t + 6 = 0 \Leftrightarrow (t+2)(t+3)$

therefore, there are three eigen values: $t_1 = 0, t_2 = -2, t_3 = -3$

to find eigen vectors:

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix} \Rightarrow \begin{cases} -x_1 + 2x_2 + 2x_3 = \lambda x_1 \\ 2x_1 + 2x_2 + 2x_3 = \lambda x_2 \\ -3x_1 - 6x_2 - 6x_3 = \lambda x_3 \end{cases}$$

$$\Rightarrow \begin{cases} 2x_2 + 2x_3 = (\lambda+1)x_1 \\ 2x_1 + 2x_3 = (\lambda-2)x_2 \\ -3x_1 - 6x_2 = (\lambda+6)x_3 \end{cases}$$

If $\lambda = 0 \Rightarrow \begin{cases} 2x_2 + 2x_3 = x_1 \\ 2x_1 + 2x_3 = -2x_2 \Rightarrow 4x_2 + 4x_3 + 2x_3 = -2x_2 \Rightarrow 6x_3 = -6x_2 \Rightarrow x_3 = -x_2 \\ -3x_1 - 6x_2 = 6x_3 \Rightarrow -3x_1 - 6x_2 = -6x_2 \Rightarrow 3x_1 = 0 \Rightarrow x_1 = 0 \end{cases}$

$$V_0 = \left\{ \begin{pmatrix} 0 \\ x_2 \\ -x_2 \end{pmatrix} \middle| x_2 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

If $\lambda = -2 \Rightarrow \begin{cases} 2x_2 + 2x_3 = -x_1 \\ 2x_1 + 2x_3 = -4x_2 \Rightarrow -4x_2 - 4x_3 + 2x_3 = -4x_2 \\ -3x_1 - 6x_2 = 4x_3 \Rightarrow -3x_1 - 6x_2 = 0 \Rightarrow -3x_1 = 6x_2 \end{cases}$ $= (*)$

$(*)\ 0 = -2x_3 \Rightarrow x_3 = 0$ $\quad x_2 = -\frac{1}{2}x_1$

$V_{(-2)} = \left\{ \begin{pmatrix} x_1 \\ -1/2 x_1 \\ 0 \end{pmatrix} \mid x_1 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix} \right\rangle$

If $\lambda = -3 \Rightarrow \begin{cases} 2x_2 + 2x_3 = -2x_1 \Rightarrow -x_2 - x_3 = x_1 \\ 2x_1 + 2x_3 = -5x_2 \Rightarrow -2x_2 - \cancel{2x_3} + \cancel{2x_3} = -5x_2 \\ -3x_1 - 6x_2 = 3x_3 \Rightarrow -3x_1 = 3x_3 \Rightarrow -x_1 = x_3 \Rightarrow x_1 = -x_3 \end{cases}$ $= (*)$

$(*)\ 0 = -3x_2 \Rightarrow x_2 = 0$

$V_{(-3)} = \left\{ \begin{pmatrix} x_1 \\ 0 \\ -x_1 \end{pmatrix} \mid x_1 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$

(e) $A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow P_A(t) = \det(A - tI) = \det \begin{pmatrix} 3-t & 2 & 1 \\ 0 & 1-t & 2 \\ 0 & 1 & -1-t \end{pmatrix}$

$= (3-t)\det\begin{pmatrix} 1-t & 2 \\ 1 & -1-t \end{pmatrix} - 2\underbrace{\det\begin{pmatrix} 0 & 2 \\ 0 & -1-t \end{pmatrix}}_{0} + 1\underbrace{\det\begin{pmatrix} 0 & 1-t \\ 0 & 1 \end{pmatrix}}_{0}$

$= (3-t)[(1-t)(-1-t) - 2] = (3-t)[-1 - \cancel{t} + \cancel{t} + t^2 - 2]$

$= (3-t)(t^2 - 3) = 3t^2 - 9 - t^3 + 3t = -t^3 + 3t^2 + 3t - 9$

<u>Eigenvalues</u>: $P_A(t) = 0 \Rightarrow -t^3 + 3t^2 + 3t - 9 = 0 \Rightarrow t^3 - 3t^2 - 3t + 9 = 0$

$= (t-3)(t^2-3) \Rightarrow t_1 = 3,\ t_2 = \sqrt{3},\ t_3 = -\sqrt{3}$

<u>Eigenvectors</u>:

$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{cases} 3x_1 + 2x_2 + 3x_3 = tx_1 \\ x_2 + 2x_3 = tx_2 \\ x_2 - x_3 = tx_3 \end{cases}$

$\Rightarrow \begin{cases} 2x_2 + 3x_3 = (t-3)x_1 \\ 2x_3 = (t-1)x_2 \\ x_2 = (t+1)x_3 \end{cases}$

If $t = 3$, $\begin{cases} 2x_2 + 3x_3 = 0 \Rightarrow 2x_2 = -3x_3 \Rightarrow x_2 = -3/2\, x_3 \\ 2x_3 = 2x_2 \Rightarrow 2x_3 = -3x_3 \Rightarrow 5x_3 = 0 \Rightarrow x_3 = 0 \\ x_2 = 4x_3 \Rightarrow x_2 = 0 \end{cases}$

$V_3 = \left\{ \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} \mid x_1 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$

If $t=\sqrt{3}$

$$\begin{cases} 2x_2+3x_3=(\sqrt{3}-3)\,x_1 \\ \quad 2x_3=(\sqrt{3}-1)\,x_2 \Rightarrow 2x_3=(\sqrt{3}-1)(\sqrt{3}+1)x_3=2x_3 \Rightarrow 2x_3=2x_3 \Rightarrow x_3=x_3 \\ x_2 \quad =(\sqrt{3}+1)\,x_3 \Rightarrow \dfrac{x_2}{\sqrt{3}+1}=x_3 \end{cases}$$

$$x_1=\frac{2x_2+3x_3}{\sqrt{3}-3}=\frac{2(\sqrt{3}+1)x_3+3x_3}{\sqrt{3}-3}=\frac{2\sqrt{3}\,x_3+2x_3+3x_3}{\sqrt{3}-3}=\frac{x_3(2\sqrt{3}+5)}{\sqrt{3}-3}$$

$V_{\sqrt{3}}=\left\langle\begin{pmatrix}\frac{x_3(2\sqrt{3}+5)}{\sqrt{3}-3}\\ (\sqrt{3}+1)\,x_3\\ x_3\end{pmatrix}\right\rangle$, likewise, replacing for $t=-\sqrt{3}$

$V_{-\sqrt{3}}=\left\langle\begin{pmatrix}x_3(-2\sqrt{3}+5)/\sqrt{3}-3\\ (-\sqrt{3}+1)x_3\\ x_3\end{pmatrix}\right\rangle.$

(f) $A=\begin{pmatrix}-1&4&-2\\-3&4&0\\-3&1&3\end{pmatrix} \Rightarrow P_A(t)=\det(A-tI)=\det\begin{pmatrix}-1-t&4&-2\\-3&4-t&0\\-3&1&3-t\end{pmatrix}$

$=(-1-t)\det\begin{pmatrix}4-t&0\\1&3-t\end{pmatrix}-4\det\begin{pmatrix}-3&0\\-3&3-t\end{pmatrix}-2\det\begin{pmatrix}-3&4-t\\-3&1\end{pmatrix}$

$=(-1-t)(4-t)(3-t)-4(-3)(3-t)-2(-3-(-3)(4-t))$

$=(-1-t)(4-t)(3-t)+36-12t-2(-3-(-12+3t))$

$=(-1-t)(4-t)(3-t)+36-12t-2(9-3t)$

$=(-1-t)(4-t)(3-t)+36-12t-18+6t$

$=(-1-t)(4-t)(3-t)+18-6t$

$=(-1-t)(12-4t-3t+t^2)+18-6t$

$=\cancel{-12}+4t+3t-t^2-12t+4t^2+3t^2-t^3+18-6t$

$=-t^3+6t^2-11t+6$

<u>Eigenvalues</u>: $P_A(t)=0 \iff -t^3+6t^2-11t+6=0 \iff t^3-6t^2+11t-6=0$

$\iff (t-1)(t-2)(t-3)=0$, the eigen values are $1, 2, 3$

<u>Eigenvectors</u>:

$$\begin{pmatrix}-1&4&-2\\-3&4&0\\-3&1&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}\lambda x_1\\ \lambda x_2\\ \lambda x_3\end{pmatrix} \Rightarrow \begin{cases}-x_1+4x_2-2x_3=\lambda x_1\\ -3x_1+4x_2 \qquad =\lambda x_2\\ -3x_1+x_2+3x_3=\lambda x_3\end{cases}$$

$$\Rightarrow \begin{cases}4x_2-2x_3=(\lambda+1)\,x_1\\ -3x_1 \qquad =(\lambda-4)\,x_2 \Rightarrow x_1=\dfrac{(\lambda-4)}{-3}\,x_2\\ -3x_1+x_2=(\lambda-3)\,x_3\end{cases}$$

If $\lambda = 1 \Rightarrow \begin{cases} 4x_2 - 2x_3 = 2x_1 \\ -3x_1 = -3x_2 \Rightarrow x_1 = x_2 \\ -3x_1 + x_2 = -2x_3 \end{cases}$

$\Downarrow$

$-3x_1 + x_1 = -2x_3 \Rightarrow -2x_1 = -2x_3$

$\Rightarrow x_1 = x_3 = x_2$

$V_1 = \left\{ \begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix} \mid x_1 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

If $\lambda = 2 \Rightarrow \begin{cases} 4x_2 - 2x_3 = 3x_1 \\ -3x_1 = -2x_2 \Rightarrow x_2 = \frac{3}{2}x_1 \\ -3x_1 + x_2 = -x_3 \end{cases}$

$\Downarrow$

$-3x_1 + \frac{3}{2}x_1 = -x_3 \Rightarrow x_1\left(-3 + \frac{3}{2}\right) = -x_3$

$-\frac{3}{2}x_1 = -x_3 \Rightarrow x_3 = \frac{3}{2}x_1 \Rightarrow$

$V_2 = \left\{ \begin{pmatrix} x_1 \\ 3/2\, x_1 \\ 3/2\, x_1 \end{pmatrix} \mid x_1 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 3/2 \\ 3/2 \end{pmatrix} \right\rangle$

If $\lambda = 3 \Rightarrow \begin{cases} 4x_2 - 2x_3 = 4x_1 \\ -3x_1 = -x_2 \Rightarrow -3x_1 = -3x_1 \Rightarrow x_1 = x_1 \\ -3x_1 + x_2 = 0 \Rightarrow -3x_1 = -x_2 \Rightarrow x_2 = 3x_1 \end{cases}$

$4(3x_1) - 2x_3 = 4x_1 \Rightarrow 12x_1 - 4x_1 = 2x_3 \Rightarrow 8x_1 = 2x_3 \Rightarrow 4x_1 = x_3$

$V_3 = \left\{ \begin{pmatrix} x_1 \\ 3x_1 \\ 4x_1 \end{pmatrix} \mid x_1 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right\rangle$.

8.2.9. V = n-dimensional v.s. $L: V \to V$ a linear map. $P_A(t)$ has n distinct roots. Show that V has a basis consisting of eigenvectors of A.

Pf: By definition, $P_A(t) = \det(A - tI)$. If $\det(A - tI) = 0$ has n distinct roots, then it looks like $(t - a_1)(t - a_2) \cdots (t - a_n) = 0$. for some a_i, possibly all equal. It suffices to show that the n eigenvectors associated with each eigenvalue (i.e, each a_i) are independent. If so, then we will have n independent vectors on an n-dimensional V.S. which imply that these are a basis.

Each root of $P_A(t)$ correspond to an eigenvalue. Each eigenvalue has a non-zero eigenvector. Let $v_1, \ldots, v_n$, (v_i) be the eigenvector with eigenvalue i. TAKE A LINEAR combination:

$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$, Apply T in both sides:

$T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = T(0)$. By linearity and the fact $T(0) = 0$

$c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n) = 0$ Eigenvector: $T(v_i) = a_i v_i \ \forall i$

$c_1 a_1 v_1 + c_2 a_2 v_2 + \cdots + c_n a_n v_n = 0$. But $v_i \neq 0 \ \forall i$, hence

$c_1 a_1 = c_2 a_2 = \cdots = c_n a_n = 0 \Rightarrow \{v_1, \ldots, v_n\}$ is independent and a basis.

8.2.10. Let A be a square matrix. Show that the eigenvalues of ${}^tA$ are the same as those of A.

Pf: By previous work (page 172) (theorem 7.5), we know that the determinat of a matrix A is equal to the determinant of its transpose. Hence,

$P_A(t) = \det(A - I\lambda) = P_{{}^tA}(t) = \det({}^tA - I\lambda)$

the eigenvalues are the roots of $P_A(t)$, which are the same roots of $P_{{}^tA}(t)$. Hence, the eigenvalues of A are the same as the eigenvalues of ${}^tA$.

8.2.11 Let A be an invertible matrix. If λ is an eigen value of A, show that $\lambda \neq 0$ and that λ^{-1} is an eigenvalue of A^{-1}.

Pf: By hypothesis, λ is an eigenvalue of A, i.e.,

$Av = \lambda v$, for some $v \neq 0$.

$A^{-1}(Av) = A^{-1}(\lambda v)$, applying A^{-1} to both sides

$(A^{-1}A)v = \lambda(A^{-1}v)$, grouping and linearity of A^{-1}

$Iv = \lambda A^{-1}v$, By definition of inverse.

$v = \lambda A^{-1}v$, dividing by λ, which we asum to be $\lambda \neq 0$

$\frac{1}{\lambda} v = A^{-1}v \Rightarrow \frac{1}{\lambda}$ is an eigen value of A^{-1}.

8.2.12. Let $V = \langle \{\sin t, \cos t\} \rangle$ over $\mathbb{R}$! Does $D: V \to V$, D is the derivative, have any non-zero eigenvectors in V? If so, which?

Solution: the matrix of the derivative with respect to $\{\sin t, \cos t\}$ is:

$D(\sin t) = \cos t = 0 \cdot \sin t + 1 \cdot \cos t$
$D(\cos t) = -\sin t = -1 \cdot \sin t + 0 \cdot \cos t$ $\Rightarrow D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

we can check that indeed $D(\sin t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \cos t$

$D(\cos t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \sin t$.

Hence, an eigenvector of D correspond with an eigen vector of this matrix:

$P_D(t) = \det(D - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1$. the eigen values satisfy:

$P_D(t) = 0 \Leftrightarrow \lambda^2 + 1 = 0$, which has no real roots.

therefore, D (the derivative) over $\mathbb{R}$ has no non-zero eigenvectors.

8.2.13. Show that the functions $\sin(kx)$ and $\cos(kx)$ are eigenvectors for D^2. What are the eigen values?

Solution: Apply D^2 to each function:

$D^2(\sin kx) = kD(\cos kx) = -k^2 \sin kx$

Hence, $\sin kx$ is an eigen vector with eigen value $-k^2$

$D^2(\cos kx) = -kD(\sin kx) = -k^2 \cos kx$

Hence, $\cos kx$ is an eigenvector with eigen value $-k^2$

8.2.15 Let A, B be square matrices of the same size. Show that the eigenvalues of AB are the same as the eigenvalues of BA.

Solution: By definition, an eigen value λ of AB is:

$(AB)v = \lambda v$ operating by B in both sides

$B((AB)v) = B(\lambda v)$ Associativity and linearity of B

$(BA)(Bv) = \lambda(Bv)$

$\Rightarrow \lambda$ is an eigen value of BA with eigenvector Bv.

If we operate instead $(BA)v = \lambda v \Rightarrow (AB)Av = \lambda(Av)$

$\Rightarrow \lambda$ is an eigen value of AB with eigenvector Av.

Additional Exercises:

1) Show that $T: V \to V$ is diagonalizable iff V is a direct sum of the eigenspaces of T.

Pf: By definition, a linear mapping $T: V \to V$ is diagonalizable iff $\exists$ basis $\{v_1, \dots, v_n\}$ of V and $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ so that $T(v_i) = \lambda_i v_i$ $\forall i = 1, \dots, n$

($\Rightarrow$) Assume T is diagonalizable, show $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_n}$

To show direct sum we need to show two properties:

(1) Show every $v \in V$ can be written as $v = \sum w_i$ where $w_i \in V_{\lambda_i}$.

Because $\{v_1, \dots, v_n\}$ is a basis, every $v \in V$ can be written as:

$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$. But, $T(v_i) = \lambda_i v_i$. Hence $\lambda_i v_i \in V_{\lambda_i}$

(2) $V_{\lambda_1} \cap V_{\lambda_2} \cap \cdots \cap V_{\lambda_n} = \{\theta\}$.

Let $z \in V_{\lambda_i}$ $\forall i = 1, \ldots, n$. then, $T(z) = \lambda_i z$ $\forall i = 1, \ldots, n$

$\Rightarrow$ Because not all λ_is are zero $\Rightarrow z = \theta$.

Hence $V = \bigoplus_{i=1}^{n} V_{\lambda_i}$.

($\Leftarrow$) Assume that $V = \bigoplus_{i=1}^{n} V_{\lambda_i}$. then $\{v_1, \ldots, v_n\}$ where $v_i \in V_{\lambda_i}$ form a basis for V. But, by definition $v_i \in V_{\lambda_1}$ iff $T(v_i) = \lambda_i v_i$. Hence, T is diagonalizable on account of the existence of the basis $\{v_1, \ldots, v_n\}$.

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(2) Let V be the v.s. of 2×2 matrices with real entries. Define $T: V \to V$ by $T(A) = B^{-1}AB$, where B is the 2×2 matrix whose first row is $[0,1]$ and whose second row is $[1,0]$. Find the eigenvalues and eigenvectors of T.

<u>Solution</u>: $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ because $BB^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

Hence, $T: V \to V$, $T(A) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

First, we need to compute the matrix associated with T.

Let $\{A_{11}, A_{12}, A_{21}, A_{22}\}$ be the basis of v.s. of 2×2 matrices, where $A_{ij} = 1$ in (i,j) position and 0 everywhere else. then,

$$T(A_{11}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = A_{22}$$

$$T(A_{12}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = A_{21}$$

$$T(A_{21}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A_{12}$$

$$T(A_{22}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A_{11}$$

$$T(A_{11}) = 0A_{11} + 0A_{12} + 0A_{21} + 1A_{22}$$
$$T(A_{12}) = 0A_{11} + 0A_{12} + 1A_{21} + 0A_{22}$$
$$T(A_{21}) = 0A_{11} + 1A_{12} + 0A_{21} + 0A_{22}$$
$$T(A_{22}) = 1A_{11} + 0A_{12} + 0A_{21} + 0A_{22}$$

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the matrix of the linear transformation T is:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = A_T$$

We want to find eigenvalues and eigenvectors of A_T, i.e.,

$$A_T(v) = \lambda v \Rightarrow P_{A_T}(t) = \det(A_T - \lambda I) = \det\begin{pmatrix} -t & 0 & 0 & 1 \\ 0 & -t & 1 & 0 \\ 0 & 1 & -t & 0 \\ 1 & 0 & 0 & -t \end{pmatrix}$$

$$= (-t)\det\begin{pmatrix} -t & 1 & 0 \\ 1 & -t & 0 \\ 0 & 0 & -t \end{pmatrix} - 1\det\begin{pmatrix} 0 & -t & 1 \\ 0 & 1 & -t \\ 1 & 0 & 0 \end{pmatrix}$$

$$= (-t)\left((-t)\det\begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix} - 1\cdot\det\begin{pmatrix} 1 & 0 \\ 0 & -t \end{pmatrix}\right) - (1)\left((t)\det\begin{pmatrix} 0 & -t \\ 1 & 0 \end{pmatrix} + 1\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

$$= (-t)\left((-t)(t^2) + t\right) - (t^2 - 1) = t^4 - t^2 - t^2 + 1 = t^4 - 2t^2 + 1$$

the eigenvalues satisfy: $P_{A_T}(t) = 0 \iff t^4 - 2t^2 + 1 = 0$

$\iff (t-1)^2(t+1)^2 = 0$. the eigenvalues are $+1$ and -1.

<u>Eigenvectors</u>:

For λ_{+1}:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow \begin{cases} w = x \\ z = y \\ y = z \\ x = w \end{cases}$$

$$V_{\lambda_{+1}} = \left\{\begin{pmatrix} x \\ y \\ y \\ x \end{pmatrix} \middle| x, y \in \mathbb{R}\right\} = \left\langle\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\}\right\rangle$$

We can check:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ y \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \\ y \\ x \end{bmatrix} = 1\cdot\begin{bmatrix} x \\ y \\ y \\ x \end{bmatrix}.$$

In terms of the 2x2 matrices, $V_{\lambda_{+1}} = \left\{\begin{pmatrix} x & y \\ y & x \end{pmatrix} \middle| x, y \in \mathbb{R}\right\}$.

We can also check this:

$$T\begin{pmatrix} x & y \\ y & x \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} x & y \\ y & x \end{pmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y & x \\ x & y \end{bmatrix} = 1\begin{bmatrix} x & y \\ y & x \end{bmatrix}$$

For λ_{-1}:

$$\begin{bmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}-x\\-y\\-z\\-w\end{bmatrix} \Rightarrow \begin{cases} w=-x\\ z=-y\\ y=-z\\ x=-w\end{cases}$$

$$V_{\lambda(-1)}=\left\{\begin{pmatrix}x\\y\\-y\\-x\end{pmatrix} \mid x,y\in\mathbb{R}\right\}=\left\langle\left\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\right\}\right\rangle.$$

we can check:

$$\begin{bmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\-y\\-x\end{bmatrix}=\begin{bmatrix}-x\\-y\\y\\x\end{bmatrix}=-1\begin{bmatrix}x\\y\\-y\\-x\end{bmatrix}$$

In terms of 2x2 matrices: $V_{\lambda(-1)}=\left\{\begin{pmatrix}x&y\\-y&-x\end{pmatrix} \mid x,y\in\mathbb{R}\right\}$

we can also check this:

$$T\begin{pmatrix}x&y\\-y&-x\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}x&y\\-y&-x\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}y&x\\-x&-y\end{pmatrix}=\begin{pmatrix}-x&-y\\y&x\end{pmatrix}$$
$$=-1\begin{pmatrix}x&y\\-y&-x\end{pmatrix}$$

---

(3) $T:\mathbb{R}^2\to\mathbb{R}^2$ $\quad T=\begin{bmatrix}0&2\\-1&0\end{bmatrix}$. Is T diagonalizeble?

<u>solution</u>:

$$P_T(\lambda)=\det(T-\lambda I)=\det\begin{pmatrix}-\lambda&2\\-1&-\lambda\end{pmatrix}=\lambda^2+2.$$

<u>Eigenvalues</u>: $P_T(\lambda)=0 \Rightarrow \lambda^2+2=0 \Rightarrow \lambda=\pm\sqrt{-2}$.

Hence, If T is viewed as $T:\mathbb{R}^2\to\mathbb{R}^2$, then T is not diagonalizable because it fails to have any eigen value.

On the contrary, if T is viewed as $T:\mathbb{C}^2\to\mathbb{C}^2$, then:

$$P_T(\lambda)=0 \Rightarrow \lambda=\pm\sqrt{2}\,i.$$

thus, we have two eigen values $\lambda_1=\sqrt{2}i$ and $\lambda_2=-\sqrt{2}i$

<u>Eigen vectors</u>:

$$\begin{bmatrix}0&2\\-1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\sqrt{2}i\begin{bmatrix}x_1\\x_2\end{bmatrix} \Rightarrow \begin{cases}2x_2=\sqrt{2}i\,x_1\\ -x_1=\sqrt{2}i\,x_2 \Rightarrow x_1=-\sqrt{2}i\,x_2.\end{cases}$$

$$V_{\lambda_1}=\left\{\begin{pmatrix}-\sqrt{2}ix_2\\x_2\end{pmatrix}\right\}=\left\langle\begin{pmatrix}-\sqrt{2}i\\1\end{pmatrix}\right\rangle.$$ We can check:

$$\begin{bmatrix}0&2\\-1&0\end{bmatrix}\begin{bmatrix}-\sqrt{2}ix_2\\x_2\end{bmatrix}=\begin{bmatrix}2x_2\\\sqrt{2}ix_2\end{bmatrix}=\sqrt{2}i\begin{bmatrix}-\sqrt{2}ix_2\\x_2\end{bmatrix}$$

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$$\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\sqrt{2}i\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{cases} 2x_2 = -\sqrt{2}i x_1 \\ -x_1 = -\sqrt{2}i x_2 \end{cases} \Rightarrow x_1 = \sqrt{2}i x_2.$$

$V_{\lambda_2} = \left\{ \begin{pmatrix} \sqrt{2}i x_2 \\ x_2 \end{pmatrix} \right\} = \left\langle \begin{pmatrix} \sqrt{2}i \\ 1 \end{pmatrix} \right\rangle$. we can check:

$$\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2}i x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ -\sqrt{2}i x_2 \end{bmatrix} = -\sqrt{2}i\begin{bmatrix} \sqrt{2}i x_2 \\ x_2 \end{bmatrix}.$$

Hence, $\exists$ a basis $\left\{ \begin{pmatrix} -\sqrt{2}i \\ 1 \end{pmatrix}, \begin{pmatrix} \sqrt{2}i \\ 1 \end{pmatrix} \right\}$ and $\lambda_1 = \sqrt{2}i, \lambda_2 = -\sqrt{2}i$, so that $T(v_i) = \lambda_i v_i$, for $i=1,2$. $\Rightarrow$ T is diagonalizable in $\mathbb{C}$.

Section 8.4.

(15). Let V be a v.s. of dimension n over $\mathbb{R}$, with a positive definite scalar product.

Let $A: V \to V$ be a symmetric linear map. Prove that the following conditions on A imply each other.

(a) $\Rightarrow$ (b). Assume that all eigenvalues of A are >0.

By the spectral theorem, V has an orthonormal basis consisting of eigenvectors. Let such a basis be $\{v_1, \dots, v_n\}$. then, for any $v \in V$, we have

$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n.$$

TAKE $\langle Av, v\rangle = \langle A(a_1v_1 + a_2v_2 + \dots + a_nv_n), a_1v_1 + a_2v_2 + \dots + a_nv_n\rangle =$

By properties of linear map A:

$= \langle a_1Av_1 + a_2Av_2 + \dots + a_nAv_n, a_1v_1 + a_2v_2 + \dots + a_nv_n\rangle =$

But v_i is an eigenvector, hence

$= \langle a_1\lambda_1v_1 + a_2\lambda_2v_2 + \dots + a_n\lambda_nv_n, a_1v_1 + a_2v_2 + \dots + a_nv_n\rangle =$

By Bilinearity

$= a_1^2\lambda_1\langle v_1,v_1\rangle + a_1a_2\lambda_1\langle v_1,v_2\rangle + \dots + a_1a_n\lambda_1\langle v_1,v_n\rangle + a_2a_1\lambda_2\langle v_2,v_1\rangle + a_2^2\lambda_2\langle v_2,v_2\rangle + \dots + a_2a_n\lambda_2\langle v_2,v_n\rangle + \dots + a_na_1\lambda_n\langle v_n,v_1\rangle + a_na_2\lambda_n\langle v_n,v_2\rangle \dots + a_n^2\lambda_n\langle v_n,v_n\rangle =$

But, $\{v_1, \dots, v_n\}$ is orthogonal $\Rightarrow \langle v_i, v_j\rangle = 0 \;\forall i \neq j$

$= a_1^2\lambda_1\langle v_1,v_1\rangle + a_2^2\lambda_2\langle v_2,v_2\rangle + \dots + a_n^2\lambda_n\langle v_n,v_n\rangle$

$\langle,\rangle$ is pos def. and $v_i \neq 0 \;\forall i \Rightarrow \langle v_i,v_i\rangle > 0$ and $a_i^2 > 0$ and, By hypothesis $\lambda_i > 0 \;\forall i$, Hence

$\langle Av, v\rangle = \sum_{i=1}^{n} a_i^2 \lambda_i \langle v_i, v_i\rangle > 0 \implies \langle Av, v\rangle > 0$

(b) $\Rightarrow$ (a). Assume that if $v \in V$, $v \neq 0$, then $\langle Av, v\rangle > 0$

Let v $(v \neq 0)$ be an eigenvector with eigenvalue λ of A then

$$Av = \lambda v \implies Av - \lambda v = 0$$

$\langle , \rangle$ is a pos. def. scalar product, hence,

$$\langle 0, 0\rangle = 0 \implies \langle Av - \lambda v, Av - \lambda v\rangle = 0$$

By bilinearity $\Rightarrow$ $\langle Av, Av\rangle - \langle Av, \lambda v\rangle - \langle \lambda v, Av\rangle + \langle \lambda v, \lambda v\rangle =$

Symmetry of $\langle , \rangle$ $\Rightarrow$ $\langle Av, Av\rangle - \lambda\langle Av, v\rangle - \lambda\langle Av, v\rangle + \lambda^2\langle v, v\rangle$

But, $Av = \lambda v$ $\quad \langle Av, Av\rangle - 2\lambda\langle Av, v\rangle + \lambda^2\langle v, v\rangle =$

$$\Rightarrow \lambda\langle Av, v\rangle - 2\lambda\langle Av, v\rangle + \lambda^2\langle v, v\rangle =$$

$$\Rightarrow \lambda^2\langle v, v\rangle - \lambda\langle Av, v\rangle = 0$$

$$\Rightarrow \lambda^2\langle v, v\rangle = \lambda\langle Av, v\rangle$$

Assuming $\lambda \neq 0$ $\Rightarrow$ $\lambda\langle v, v\rangle = \langle Av, v\rangle$

By hypothesis $\langle Av, v\rangle > 0$ and $\langle v, v\rangle > 0$ because $v \neq 0$

$\Rightarrow \lambda > 0$ for all eigenvalues of A.

(8.4.24) Let V be a v.s. of dimension n over $\mathbb{R}$, with a positive definite scalar product. Let $A: V \to V$ be a symmetric operator.

To show A has only one eigenvalue, suppose A has two distinct eigenvalues λ_1 and λ_2. Assuming that A has no invariant subspaces other than Θ and V, we get a contradiction because V_{λ_1} is such that if $v \in V_{\lambda_1}$ then $Av \in V_{\lambda_1}$, hence V_{λ_1} is stable under A and so is V_{λ_2}. Hence, A has only one eigenvalue. then, using the Spectral theorem, we have an orthonormal basis $\{v_1, \dots, v_n\}$ for V.

HAVE $\quad Av = A\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} a_i A v_i = \sum_{i=1}^{n} a_i \lambda v_i = \lambda\sum_{i=1}^{n} a_i v_i = \lambda v$

$\Rightarrow A = \lambda I$ because $\lambda I v = \lambda v$.

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Additional Exercises:

1) $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. $\Rightarrow P_A(t) = \det(A - tI) = \det\begin{pmatrix} 2-t & -1 \\ -1 & 2-t \end{pmatrix}$

$= (2-t)^2 - 1 = 4 - 4t + t^2 - 1 = t^2 - 4t + 3 = (t-3)(t-1)$

$\Rightarrow$ the eigenvalues of A are $\lambda_1 = 3$, $\lambda_2 = 1$.

The critical values of $f(x) = \dfrac{\langle Ax, x\rangle}{\langle x, x\rangle}$ are; by theorem in class, precisely the eigenvalues 3, 1.

We can check this using calculus:

$$f(x) = \frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \frac{\left\langle \begin{pmatrix} 2 & -1 \\ -1 & 2\end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\rangle}{\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\rangle} = \frac{\left\langle \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\rangle}{x_1^2 + x_2^2}$$

$\Rightarrow f(x) = \dfrac{2x_1^2 - 2x_1x_2 + 2x_2^2}{x_1^2 + x_2^2}$. To find critical points, set

$\nabla f = 0 \Rightarrow \left(\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}\right) = (0, 0)$.

$$\frac{\partial f}{\partial x_1} = \frac{(4x_1 - 2x_2)(x_1^2 + x_2^2) - (2x_1^2 - 2x_1x_2 + 2x_2^2)(2x_1)}{(x_1^2 + x_2^2)^2} = 0$$

$\Rightarrow (4x_1 - 2x_2)(x_1^2 + x_2^2) - (2x_1^2 - 2x_1x_2 + 2x_2^2)(2x_1) = 0$

$= (4x_1^3 + 4x_1x_2^2 - 2x_1^2x_2 - 2x_2^3) - (4x_1^3 - 4x_1^2x_2 + 4x_1x_2^2)$

$= \cancel{4x_1^3} + \cancel{4x_1x_2^2} - 2x_1^2x_2 - 2x_2^3 - \cancel{4x_1^3} + 4x_1^2x_2 - \cancel{4x_1x_2^2}$

$= 2x_1^2x_2 - 2x_2^3 \Rightarrow \boxed{\dfrac{\partial f}{\partial x_1} = 0 = x_1^2x_2 - x_2^3}$

$\Rightarrow x_2^3 = x_1^2x_2$, the candidate points are $(0,0), (1,1), (-1,1)$ $(1,-1)$ and $(-1,-1)$

Likewise,

$$\frac{\partial f}{\partial x_2} = \frac{(-2x_1 + 4x_2)(x_1^2 + x_2^2) - (2x_1^2 - 2x_1x_2 + 2x_2^2)(2x_2)}{(x_1^2 + x_2^2)^2} = 0$$

$\Longrightarrow$

$= (-2x_1^3 - 2x_1x_2^2 + 4x_1^2x_2 + 4x_2^3) - (4x_1^2x_2 - 4x_1x_2^2 + 4x_2^3)$

$= -2x_1^3 - 2x_1x_2^2 \cancel{+ 4x_1^2x_2} \cancel{+ 4x_2^3} \cancel{- 4x_1^2x_2} + 4x_1x_2^2 \cancel{- 4x_2^3}$

$= -2x_1^3 + 2x_1x_2^2$

$\Rightarrow \boxed{\frac{\partial f}{\partial x_2} = 0 = x_1x_2^2 - x_1^3} \Rightarrow x_1^3 = x_1x_2^2$.

the condidate points are $(0,0), (1,1), (-1,1), (1,-1)$ and $(-1,-1)$

Exactly the same as for $\frac{\partial f}{\partial x_1}$.

to find critical values, we plug the condidate points back into $f(x)$:

$f(0,0)$ is not defined.

$f(1,1) = \frac{2-2+2}{2} = 1 = f(-1,-1) \Rightarrow$ correspond to eigenvalue 1

$f(-1,1) = \frac{2+2+2}{2} = 3 = f(1,-1) \Rightarrow$ correspond to eigenvalue 3

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(2) $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$, the critical values are given by:

$P_A(t) = 0 \Rightarrow \det(A - tI) = \det\begin{pmatrix} 1-t & -1 & 0 \\ -1 & 2-t & -1 \\ 0 & -1 & 1-t \end{pmatrix}$

$= (1-t)\det\begin{pmatrix} 2-t & -1 \\ -1 & 1-t \end{pmatrix} + \det\begin{pmatrix} -1 & -1 \\ 0 & 1-t \end{pmatrix}$

$= (1-t)[(2-t)(1-t) - 1] + t - 1 = (1-t)[2 - 2t - t + t^2 - 1] + t - 1$

$= (1-t)(t^2 - 3t + 1) + t - 1 = t^2 - 3t + 1 - t^3 + 3t^2 - t + t - 1$

$= -t^3 + 4t^2 - 3t$

Eigenvalues: $P_A(t) = 0 \Leftrightarrow -t^3 + 4t^2 - 3t = 0 \Leftrightarrow t^3 - 4t^2 + 3t = 0$

$t(t^2 - 4t + 3) = 0 \Leftrightarrow t(t-1)(t-3) = 0$

Hence, the eigenvalues are $t = 0$, $t = 1$ and $t = 3$.

By previous theorem, these are the critical values for $f(x) = \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$, where $x \in \mathbb{R}^3$.

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Section 8.3.

(3) Find the maximum and minimum of the function

$$f(x,y) = 3x^2 + 5xy - 4y^2$$

on the unit circle.

Solution: First, find the matrix associated to f; i.e.,

$$(x \ \ y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3x^2 + 5xy - 4y^2$$

$$\Rightarrow (x \ \ y)\begin{pmatrix} ax+by \\ bx+dy \end{pmatrix} = 3x^2 + 5xy - 4y^2$$

$$\Rightarrow ax^2 + 2bxy + dy^2 = 3x^2 + 5xy - 4y^2$$

$$\Rightarrow a = 3, \ b = \frac{5}{2}, \ d = -4.$$

We obtain the matrix

$A = \begin{pmatrix} 3 & 5/2 \\ 5/2 & -4 \end{pmatrix}$. To find its critical values, we compute its eigenvalues:

$$P_A(t) = \det(A - tI) = \det\begin{pmatrix} 3-t & 5/2 \\ 5/2 & -4-t \end{pmatrix} = (3-t)(-4-t) - \frac{25}{4}$$

$$= -12 - 3t + 4t + t^2 - \frac{25}{4} = t^2 + t - \frac{73}{4}$$

Eigenvalues $\Rightarrow t^2 + t - \frac{73}{4} = 0$. Using quadratic solver:

$\frac{-1 \pm \sqrt{1+73}}{2} = \frac{-1 \pm \sqrt{74}}{2}$, Hence, the eigenvalues are $\frac{-1-\sqrt{74}}{2}$, $\frac{-1+\sqrt{74}}{2}$.

the maximum is $\frac{-1+\sqrt{74}}{2}$, the minimum is $\frac{-1-\sqrt{74}}{2}$.

Additional Exercise Corrected:

(a) the bijection $\psi : A \to A'$, where A and A' satisfy (1) and (2) can be defined as:

$$\psi(i,j) = \begin{cases} (i,j) & \text{if } (i,j) \in A' \\ (j,i) & \text{if } (i,j) \notin A' \end{cases}$$

this is a bijection. Proof:

(1) ψ is injective: Let $\psi(i,j) = \psi(k,l)$. By definition of ψ, if $(i,j) \in A'$, then $\psi(i,j) = (i,j) = \psi(k,l) \Rightarrow (i,j) = (k,l)$

If $(i,j) \notin A'$, then $\psi(i,j) = (j,i) = \psi(k,l) \Rightarrow (i,j) = (k,l)$.

A symmetric case occurs for $(k,l) \in A'$ and $(k,l) \notin A'$.

(2) ψ is surjective: $\forall (k,l) \in A'$. $\exists (i,j) \in A$ s.t. $\psi(i,j) = (k,l)$

Let $(k,l) \in A'$. then, By definition of ψ, and conditions on A, A', either $(k,l) \in A$, in which case $\psi(k,l) = (k,l)$, OR $(l,k) \in A$, in which case $\psi(l,k) = (k,l)$.

(1) & (2) $\Rightarrow$ Bijectivity.

(b) f is a permutation of $\{1, 2, \ldots, n\}$.

$\prod_{(i,j)\in A} \frac{x_{f(i)} - x_{f(j)}}{x_i - x_j} =$ using the bijection ψ, we have two posibilities, either we are dealing with the identity, in which case is true that:

$$\prod_{(i,j)\in A} \frac{x_{f(i)} - x_{f(j)}}{x_i - x_j} = \prod_{(i,j)\in A'} \frac{x_{f(i)} - x_{f(j)}}{x_i - x_j}$$

Or, we are dealing with the case in which $\psi(i,j) = (j,i)$, in which case:

$$\prod_{(i,j)\in A} \frac{x_{f(i)} - x_{f(j)}}{x_i - x_j} = \prod_{\psi(i,j)\in A} \frac{x_{f(j)} - x_{f(i)}}{x_j - x_i} = \prod_{(j,i)\in A} \frac{(-1)\,x_{f(j)} - x_{f(i)}}{(-1)\,x_j - x_i}$$

$$= \prod_{(j,i)\in A} \frac{x_{f(i)} - x_{f(j)}}{x_i - x_j}$$

(C) if g is another permutation.

$$\prod_{i<j}\frac{x_{f(g(i))}-x_{f(g(j))}}{x_{g(i)}-x_{g(j)}} = \prod_{i<j}\frac{x_{f(g(\tilde{\varphi}(i)))}-x_{f(g(\tilde{\varphi}(j)))}}{x_{g(\tilde{\varphi}(i))}-x_{g(\tilde{\varphi}(j))}} = (*)$$

where $\tilde{\varphi}(i)=$ the component i of $\varphi(i,j)$,

Hence, If we are on case 1 of $\varphi(i,j)$, then

$$(*) = \prod_{i<j}\frac{x_{f(i)}-x_{f(j)}}{x_i-x_j}$$

Otherwise, If we are on case 2 of φ, then

$$(*) = \prod_{i<j}\frac{x_{f(j)}-x_{f(i)}}{x_j-x_i} =$$ multiply and divide by -1

$$= \prod_{i<j}\frac{(-1)(x_{f(j)}-x_{f(i)})}{(-1)\,x_j-x_i} = \prod_{i<j}\frac{x_{f(i)}-x_{f(j)}}{x_i-x_j}.$$

which shows what we wanted to show.