Exam 1 M409 Summer 2012 C. Judge

NAME: Solutions

Show all work! Each problem is worth 10 points.

(1) Let V and W be vector spaces over a field K. Complete the following definitions

(a) A set $\{v_1, v_2, \ldots, v_k\} \subset V$ is linearly independent iff

$a_1 v_1 + \cdots + a_k v_k = 0$ for some $a_1, \ldots, a_k \in K$ implies that $a_1 = a_2 = \cdots = a_k = 0$.

(b) A set $\{v_1, v_2, \ldots, v_k\}$ generates V iff

for each $v \in V$, there exist $a_1, \ldots, a_k \in K$ so that $v = a_1 v_1 + \cdots + a_k v_k$.

(c) The kernel of a linear mapping $F: V \to W$ is the set

$\{v \in V \mid F(v) = 0\}$

(d) The image of a linear map $F: V \to W$ is the set

$$\{ F(v) \mid v \in V \}$$

(e) The linear mapping $L_A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ associated to an $m \times n$ matrix A is defined by

$$L_A(v) = A \cdot v$$

(2) Find a basis for the vector space of 2×2 matrices A that satisfy ${}^tA = -A$ Justify your answer.

I claim that $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ is a basis.

Indeed, $a \in K$ & $a \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies a \cdot 1 = 0 \Rightarrow a = 0$

and thus independent.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and ${}^tA = -A$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix} \implies \left. \begin{matrix} a = -a \\ d = -d \\ b = -c \end{matrix} \right\} \Rightarrow \begin{matrix} a = 0 \\ d = c \\ b = -c \end{matrix}$$

and so $A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Therefore $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ generates.

Since its independent and generates, its a basis.

(3) Let A be an $n\times n$ matrix such that $A^2 = O$.
Show that $I-A$ is invertible.

$$(I+A)(I-A) = I^2 - IA + AI - A^2 = I + A - A - \overset{\text{hypothesis}}{0} = I$$

$$(I-A)(I+A) = I^2 + IA - AI - A^2 = I + A - A - 0 = I$$

Hence $I+A$ is an inverse for $I-A$.
Thus $I-A$ is invertible.

(4) Let P be the set of all real polynomials of one variable x (i.e. $a_0 + a \cdot x + \cdots + a_n \cdot x^n$). Show that P is a vector subspace of the vector space of all real-valued functions on the real line.

(1) The zero function defined by $f(x) = 0 \ \forall x$ is a polynomial. Indeed, take the coefficients to all be zero.

(2) If $p(x) = a_0 + a_1 x + \cdots + a_k x^n$
and $q(x) = b_0 + b_1 x + \cdots + b_n x^n$
then $(p+q)(x) = p(x) + q(x)$
$(a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$
and hence $p+q$ is a polynomial

(3) If $s \in K$, then $s \cdot p(x) = s(a_0 + a_1 x + \cdots + a_n x^n)$
$= (s a_0) + (s a_1)x + \cdots + (s a_n)x^n$
and hence $s \cdot p$ is polynomial.

(5) Let V be a vector space over K.
Use the definition of vector space to show that $V \times V$ is a vector space over K where addition is defined by

$$(v,w)+(v',w') = (v+v', w+w')$$

and scalar multiplication is defined by

$$s(v,w) = (sv, sw).$$

Need to check (VS1)–(VS8)

(VS1) $((v,w)+(v',w'))+(v'',w'') = (v+v', w+w') + (v'',w'')$

$= ((v+v')+v'', (w+w')+w'')$

Since V is vector space $\longrightarrow$ $= (v+(v'+v''), w+(w'+w''))$

$= (v,w) + (v'+v'', w'+w'')$

$= (v,w) + ((v', v'') + (w', w''))$

(VS2) Since V is vector space $\exists\, 0 \in V$ s.t. $0+u = u = u+0\ \forall u$

$(0,0)+(v,w) = (0+v, 0+w) = (v,w)$

$(v,w)+(0,0) = (v+0, w+0) = (v,w)$

(VS3) Given (v,w). Since V is vector space $\exists\, -v\ \&\ -w \in V$ so that $v+(-v)=0$ and $w+(-w)=0$.

Then $(v,w)+(-v,-w) = (v+-w, v+-w) = (0,0)$

(VS4) $(v,w)+(v',w') = (v+v', w+w')$

$= (v'+v, w'+w)$ ← since V is v.s.

$= (v',w')+(v,w)$

(VS5) $c \in K \Rightarrow c((v,w)+(v',w')) = c(v+v', w+w')$

$= (c(v+v'), c(w+w'))$

Since V is vector space $\longrightarrow$ $= (cv+cv', cw+cw')$

(VS6) if $a,b \in K$, then

$(a+b)(v,w) = ((a+b)v, (a+b)w)$

$= (av+bv, aw+bw)$ ← since V is v.s.

$$= (av, aw) + (bv, bw)$$
$$= a(v,w) + b(v,w)$$

(VS7) if $a, b \in K$, then

since V is v.s.

$$(ab)(v,w) = ((ab)v, (ab)w)$$
$$= (a(bv), a(bw))$$
$$= a(bv, bw)$$
$$= a(b(v,w))$$

(VS8) $1 \cdot (v,w) = (1v, 1w) = (v,w)$

since V is a vector space

(6) Compute the matrix of $id_{\mathbb{R}^2} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with respect to the basis $\mathcal{B} = \{(0,1), (1,-1)\}$ for the domain and the basis $\mathcal{B}' = \{(1,1), (-1,0)\}$ for the codomain.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a_{11}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_{12}\begin{pmatrix} -1 \\ 0 \end{pmatrix} \Longrightarrow \quad a_{11} = 1 \quad a_{12} = 1$$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = a_{21}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_{22}\begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow \quad a_{21} = -1 \quad a_{22} = -2$$

$$-a_{22} + -1 = 1$$

$$a_{22} = -2$$

$$M^{B}_{B'}(id_{\mathbb{R}^2}) = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$$

(7) Let $T : \mathbb{R} \longrightarrow \mathbb{R}^k$ be a linear mapping. What can the dimension of the image of T be? Justify your answer.

Since $\mathbb{R}$ and $\mathbb{R}^k$ are finite dimensional

$$1 = \dim(\mathbb{R}) = \dim(\ker(T)) + \dim(\mathrm{Im}(T))$$

Since the dimension of a subspace is at most the dimension of the space. We have either $\dim(\ker(T)) = 0$ or 1

Hence $\dim(\mathrm{Im}(T))$ is either 1 or 0 by the formula above.

(8) Let $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ denote the linear transformation that rotates each vector counterclockwise by an angle of θ radians.

(a) Describe the linear transformation $R_\alpha \circ R_\beta$.

$R_\alpha \circ R_\beta$ rotates each vector counterclockwise by an angle of $\beta + \alpha$.

(b) Is the set $\{R_\theta \mid \theta \in \mathbb{R}\}$ a vector subspace of $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$? Explain why or why not.

No, for example, the zero function is not an element of $\{R_\theta \mid \theta \in \mathbb{R}\}$

(9) Let $T: \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ be defined by $T(x) = A \cdot x$ where

$$A = \begin{pmatrix} 0 & 2 & 1 & 4 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 1 & 1 \end{pmatrix}$$

What is the dimension of the kernel of T? Justify your answer.

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad T\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad T\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and hence the basis $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a subset of $Im(T)$.

In particular $Im(T) = \mathbb{R}^3$ and so $\dim(Im(T)) = 3$

But $5 = \dim(\mathbb{R}^5) = \dim(\ker(T)) + \dim(Im(T))$

and so $\dim(\ker(T)) = 2$.

(10) Let V be the vector space of all functions $f: \mathbb{R} \longrightarrow \mathbb{R}$. Define $G: V \longrightarrow V$ by

$$(G(f))(x) = f(x^2) \quad \forall x \in \mathbb{R}$$

(a) Is G linear? Explain why or why not.

Yes, let $s,t \in \mathbb{R}$ and let f and g belong to V

$$\begin{aligned} G(sf+tg)(x) &= (sf+tg)(x^2) \\ &= s f(x^2) + t g(x^2) \\ &= s G(f)(x) + t G(g)(x) \end{aligned}$$

(b) Is G injective? Explain or why not.

No, let $f(x) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0 \end{cases}$

and $g(x) = 0 \quad \forall x$

Then $f(x^2) = 0 = g(x^2)$ for all (since $x^2 \geq 0$)
but $f \neq g$.

(c) Is G surjective? Explain why or why not.

No, for example, suppose that the function f defined by $f(x) = x$ belonged to the image
Then there would exist g so that
$\forall x$. $G(g)(x) = f(x) = x$
But then $1 = f(1) = G(g)(1) = g(1^2) = g(1)$
and $-1 = f(-1) = G(g)(-1) = g((-1)^2) = g(1)$
Thus $-1 = 1$ and hence we have a contradiction