Exam 2   M409   Summer 2012   C. Judge

NAME: ______________________________

Show all work! Let K be a field and V a vector space
Each problem is worth 10 points.

(1) Complete of the following statements:

(a) A bilinear form B in positive definite if and only if

$B(v,v) > 0 \quad \forall \; v \neq 0$

(b) A set $\{v_1, v_2, v_3, \ldots, v_j\}$ is an orthogonal set if and only if

$B(v_i, v_j) = 0 \quad \forall \; i \neq j$

(c) Let $B : V \times V \longrightarrow K$ be a bilinear form. The vectors $v, w \in V$ are said to be orthogonal with respect to B if and only if

$B(v,w) = 0$

(d) A function $F: K^n \times \cdots K^n \longrightarrow K$ is an alternating multilinear form if and only if

$\forall j$ $\quad F(v_1, \ldots, v_{j+1}, v_j, \ldots, v_n) = -F(v_1, \ldots, v_j, v_{j+1}, \ldots, v_n)$

$F(v_1, \ldots, s v_j + s' v_j', \ldots, v_n) = s F(v_1, \ldots, v_j, \ldots, v_n)$
$\qquad + s' F(v_1, \ldots, v_j', \ldots, v_n)$

(e) A bijection $f: \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ is called a permutation if and only if

I exist

(2) Let $B: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the bilinear form defined by
$$B((x_1, x_2), (x_1', x_2')) = x_1 x_1' + 2 x_1 x_2' - 2 x_2 x_1' + x_2 x_2'$$

(a) Is B positive definite? Explain why or why not

Yes, $B((x_1, x_2), (x_1, x_2)) = x_1^2 + x_2^2 > 0 \quad \forall \quad (x_1, x_2) \neq (0,0)$

(b) Is B symmetric? Explain why or why not

No, $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ is not a symmetric matrix

(3) Let V be the vector space of continuous functions $f:[0,1] \longrightarrow \mathbb{R}$. Define $B: V\times V \longrightarrow \mathbb{R}$ by $B(f,g) = \int_0^1 f(t)\,g(t)\,dt$. Verify that B is a bilinear form.

$$\int_0^1 f(t)\,g(t)\,dt = \int_0^1 g(t)f(t)\,dt \implies B(f,g) = B(g,f)$$

Hence suffices to prove linearity in the first "slot"

$$\int_0^1 (sf(t) + s'h(t))g(t)\,dt = s\int_0^1 f(t)\,g(t)\,dt + s'\int_0^1 h(t)g(t)\,dt$$

$$B(sf+s'h, g) = sB(f,g) + s'B(h,g)$$

(4) Let $B: V\times V \longrightarrow \mathbb{R}$ be as in (3). Find an orthogonal basis for the vector space generated by the functions t and t^3.

$$v_1 = t \qquad v_2 = t^3 - \frac{B(t^3,t)}{B(t,t)}\,t$$

$$B(t^3,t) = \int_0^1 t^4\,dt = \frac{1}{5}t^5\Big|_0^1 = \frac{1}{5}$$

$$= t^3 - \frac{1/5}{1/3}\,t$$

$$= t^3 - \tfrac{3}{5}t$$

$\{v_1, v_2\}$ is orthogonal basis

$$B(t,t) = \int_0^1 t^2\,dt = \frac{1}{3}t^3\Big|_0^1 = \tfrac{1}{3}$$

(5) Let $B: \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the bilinear form associated to the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(a) Compute the orthogonal projection of $(0,1,0)$ onto the line spanned by $(0,0,1)$.

$$\text{projection} = \frac{B(e_2, e_3)}{B(e_3, e_3)} e_3 = \frac{1}{1} e_3 = e_3$$

(b) Find an orthogonal basis for B. (Show work)

$u_1 = \boxed{e_3}$ $\quad u_2 = e_2 - P_{\langle u_1 \rangle}(e_2) = \boxed{e_2 - e_3}$

$$u_3 = e_1 - P_{\langle u_1 \rangle}(e_1) - P_{\langle u_2 \rangle}(e_1)$$

$$= e_1 - \frac{\cancel{B(e_1, e_3)}^{\,0}}{B(e_3, e_3)} e_3 - \frac{B(e_1, e_2 - e_3)}{B(e_2 - e_3, e_2 - e_3)}(e_2 - e_3)$$

$$= e_1 - 0 - \left(\frac{B(e_1, e_2) - B(e_1, e_3)}{B(e_2, e_2) - 2B(e_2, e_3) + B(e_3, e_3)} \right)(e_2 - e_3)$$

$$= e_1 - \left(\frac{1 - 0}{-1 - 2 \cdot 1 + 1} \right)(e_2 - e_3) = \boxed{e_1 + \tfrac{1}{2} e_2 - \tfrac{1}{2} e_3}$$

(c) What is the signature of B?

$B(u_1, u_1) = 1 > 0$

$B(u_2, u_2) = -2 < 0$

$\boxed{\text{Signature} = (0, 2, 1)}$

$$B(e_1 + \tfrac{1}{2} e_2 - \tfrac{1}{2} e_3,\ e_1 + \tfrac{1}{2} e_2 - \tfrac{1}{2} e_3)$$

$$= 0 + 2\tfrac{1}{2} \cdot 1 + 2(-\tfrac{1}{2} \cdot 1) + \tfrac{1}{4}(-1) - \tfrac{2}{4} 1 + \tfrac{1}{4} 1$$

$$= 1 + \cancel{1} \cancel{-1} - \cancel{\tfrac{1}{4}} - \tfrac{1}{2} + \cancel{\tfrac{1}{4}} = \tfrac{1}{2} > 0$$

(6) (a) List all of the permutations of $\{1,2,3\}$

$id = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ $\quad \tau_{12} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ $\quad \tau_{23} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$ $\quad \tau_{13} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

$\sigma_1 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ $\quad \sigma_2 \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

(b) Determine the sign of each permutation listed above (Show work)

$\varepsilon(id) = 1$

$\varepsilon(\tau_{ij}) = -1$ since transpositions

$\sigma_1 = \tau_{12} \circ \tau_{23} \implies \varepsilon(\sigma_1) = (-1)^2 = +1$

$\sigma_2 = \tau_{13} \circ \tau_{23} \implies \varepsilon(\sigma_2) = +1$

(c) Use (a) and (b) to write the expansion formula for $D: \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$.

$$D\left(\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}\right) = \sum_{f \in Aut(\{1,2,3\})} \varepsilon(f)\, a_{1f(1)}\, a_{2f(2)}\, a_{3f(3)}$$

$$= \varepsilon(id)\, a_{11} a_{22} a_{33} + \varepsilon(\tau_{12})\, a_{12} a_{21} a_{33} + \varepsilon(\tau_{23})\, a_{11} a_{32} a_{23} + \varepsilon(\tau_{13})\, a_{13} a_{22} a_{33} + \varepsilon(\sigma_1)\, a_{12} a_{23} a_{31} + \varepsilon(\sigma_2)\, a_{13} a_{32} a_{21}$$

$$= a_{11} a_{22} a_{33} - a_{12} a_{21} a_{33} - a_{11} a_{32} a_{23} - a_{13} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{32} a_{21}$$

(7) Let $f = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$

(a) Factorize f into transpositions

Let $\tau_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{bmatrix}$ then $\tau_1 \circ f = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix}$

Let $\tau_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{bmatrix}$ then $\tau_2 \circ \tau_1 \circ f = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{bmatrix}$

Let $\tau_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{bmatrix}$ then $\tau_2 \circ \tau_1 \circ f = \tau_3$

$\Longrightarrow f = \tau_1 \circ \tau_2 \circ \tau_3$

where each τ_i is transposition.

(b) What is the sign of f? Why?

$$\varepsilon(f) = \varepsilon(\tau_1 \circ \tau_2 \circ \tau_3) = \varepsilon(\tau_1)\varepsilon(\tau_2)\varepsilon(\tau_3) = (-1)(-1)(-1) = -1$$

since τ_i transpositions.

(c) Compute $D(e_2, e_3, e_5, e_4, e_1, e_6)$. Justify.

$$\varepsilon\left(\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 4 & 1 & 6 \end{bmatrix}\right) = \varepsilon\left(\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & 1 \end{bmatrix}\right)$$

$$= \varepsilon\left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}\right) = -1$$

$$\Longrightarrow D(e_2, e_3, e_5, e_4, e_1, e_6) = -1 \cdot D(e_1, e_2, \dots, e_6)$$

$$= -1$$

(8) Let $F:(K^n)^n \longrightarrow K$ be an alternating multilinear form. Show that there exists $c \in K$ so that $cF = D$. (If you use a theorem, then please give a precise statement of the theorem.)

Theorem: The space of alternating multilinear forms $F:(K^n)^n \longrightarrow K$ is 1-dimensional

Hence F and D differ by a constant.

(9) Let $B: V \times V \longrightarrow \mathbb{R}$ be a symmetric bilinear form. Show that if $B(v,v)=0$ for all $v \in V$, then $B(v,w)=0$ for all $v,w \in V$.

$$\cancelto{0}{B(v+w,v+w)} = \cancelto{0}{B(v,v)} + B(v,w) + B(w,v) + \cancelto{0}{B(w,w)}$$

$$\Longrightarrow 0 = B(v,w) + B(w,v)$$

Since B symmetric, $0 = B(v,w) + B(v,w)$
$\Longrightarrow B(v,w)=0.$

(10) Let $B: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a symmetric bilinear form. Let $\{v_1, \ldots, v_n\}$ be orthogonal set.

(a) Is $D(v_1, v_2, \ldots, v_n)$ necessarily equal to zero? Discuss why or why not.

No, for example $D(e_1, \ldots, e_n) = 1 \neq 0$

(b) Same question but now assume that B is nondegenerate.

No, for example $D(e_1, \ldots, e_n) = 1 \neq 0$